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THIRD ORDER ITERATIVE METHOD FOR SOLVING NON-LINEAR PARABOLIC PARTIAL DIFFERENTIAL EQUATION IN FINANCIAL APPLICATION

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ABSTRACT

In this paper, a third-order iterative scheme is presented for searching approximate solutions of a non-linear parabolic partial differential equation to simulate the elaboration of interest rates in the fanatical application. First, by using Taylor series expansion we gain the discretization scheme for the model problem. Then, using the Gauss-Seidel iterative scheme we solve the proposed model problems. To validate the convergences of the proposed numerical techniques, three model illustrations are considered. The convergent analysis of the present techniques is worked by supporting the theoretical and fine numerical statements. The accuracy of the present numerical techniques has been measured by using average absolute error root mean square error and point-wise maximum absolute error. Then, we compare these get crimes with the result attained in the literature. These results are also presented in tables and graphs. The comparison of physical behavior between present numerical versus its exact solutions is also presented in terms of graphs. As we can see from the table and graphs, the present numerical techniques approximate the exact result veritably well. So, it is relatively effective for simulating fanatical application to the non-linear parabolic partial differential equation.

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1. INTRODUCTION

Partial differential equations arise not only from subfields within mathematics such as differential geometry and analysis but also from almost every scientific and engineering field as mathematical models of various application problems Feng et al. (2013). As the behavior of the solutions underlying, these application problems depend on governing partial differential equations. So, Solving, analyzing, and implementing the solution to these partial differential equations has been critically important for the resolutions of many scientific and engineering application problems. Concerning different criteria, partial differential equations can be categorized into several types.

However, using nonlinearity as a criterion, partial differential equations can be divided into two categories: linear partial differential equations and nonlinear partial differential equations. Evans (1998), Taylor (1996),Taylor (1996) . In the nonlinear category, partial differential equations are further classified as semi-linear partial differential equations, quasi-linear partial differential equations, and fully non-linear partial differential equations based on the degree of the non-linearity. Semi-linear partial differential equations are differential equation that is non-linear in the unknown function but linear in all its partial derivatives. The non-linear Poisson equation is a well-known example of this class of partial differential equations. A quasi-linear partial differential equation is non-linear in at least one of the lower order derivatives but linear in the highest order derivative of the unknown function Feng et al. (2013).

Fully non-linear second-order partial differential equations arise from many fields in science and engineering such as astrophysics, antenna design, differential geometry, geotropic fluid dynamics, image processing, materials science Ambrosio et al. (2001), mathematics and finance Ambrosio et al. (2001), Aliyi et al. (2021) mesh generation, meteorology, optimal transport, and stochastic control Ambrosio et al. (2001). Various phenomena and applications Pardoux (2007), Prevot and Rockner (2007) and the references therein with stochastic influence in natural or artificial complex systems can be modelled by Stochastic partial differential equations, including stochastic quantization of the free Euclidean quantum field, turbulence, population dynamics and genetics, neurophysiology, the evolution of the curve of interest rate, non-linear filtering, movement by mean curvature in random environment, the hydrodynamic limit of particle systems, fluctuations of an interface on a wall, and path wise stochastic control theory Li et al. (2021), Alharbi (2020). In these fundamental applications, several examples of canonical Stochastic partial differential equations arise, such as the Zakai equation, reflected stochastic heat equation, stochastic reaction-diffusion equations, stochastic Burger's equation, stochastic Navier–Stokes equation, stochastic porous media equation Li et al. (2021), and non-linear advection-diffusion equation. A common example of diffusion is given by heat conduction in a solid body Aliyi et al. (2021), Ahmed (2017). Conduction comes from molecular collision, transferring heat by kinetic energy, without macroscopic material movement Ahmed (2017). The application of non-linear partial differential equations is also found as the Black–Scholes model, see in Alharbi (2020).

The stochastic discrimination equation for the CIR garrulousness satisfies the Yamada-Watanabe condition, so it admits a unique strong result Gatheral and Taleb (2013), Rouah (n.d.). In fine finance, the Cox – Ingersoll – Ross (CIR) model describes the elaboration of interest rates. It's a type of short-rate model as it describes interest rate movements as driven by only one source of request trouble Orlando et al. (2018), Orlando et al. (2019). The model can be used in the evaluation of interest rate derivatives. A CIR process is a special case of an introductory affine jump-long-windedness, which still permits an unrestricted- form expression for bond prices. Time-varying functions replacing portions can be introduced in the model to make it harmonious with the assigned term structure of interest rates and possibly volatilities Orlando et al. (2018),. Also, non-linear equations appear in condensed matter, solid-state medicines, fluid mechanics, chemical kinetics, tube medicines, non-linear optics, propagation of fluxions in Josephson junctions, the proposition of turbulence, ocean dynamics, biophysics star evidence, and multitudinous others Maher et al. (2013). This non-linear equation has its own either exact or numerical result and these results show the behaviour of governing equation in the result intervals.

In recent years, directly searching either exact or numerical solutions of non-linear parabolic equations has become more and more attractive parts in different branches of physics and applied mathematics. The majority of non-linear parabolic partial differential equations do not have analytical solutions. But also, some numerical methods have a slow rate of convergence, instability, low accuracy, and difficulty in applying it to implement the simulation of non-linear parabolic partial differential equations in complex geometries. Therefore, due to this reason, several numerical methods have been developed for investigating the simulation of a non-linear parabolic equation. For instance, in Nhan et al. (2021), the high-order iterative scheme was used for the study of the non-linear pseudo-parabolic equation. They apply the Faedo-Galerkin approximation method and use basic concepts of non-linear analysis, but grid generation is usually more automatic for the Galerkin approximation method, although not completely for complex geometries. Also, this iterative scheme is used to search the solution of stochastic parabolic equations and study the existence of these solutions in Ngoc et al. (2010), Truong et al. (2009), Ahmed (2017). Thus, they get a better approximation of their applied governing problem.

The stochastic parabolic original value problem also mainly shops and freckled to the operation of financial models of the stochastic volatility model. The nontrivial point of the equation appears from the non-linear first-order term in spatial variable and a Holder, yet not Lipchitz, too rough to be differentiable in space and time. Indeed, still, with dropping the quadratic non-linearity from these parabolic partial discriminative equations, it's reduced to the stochastic heat equation with the mean-returning term, whose result is not differentiable Li et al. (2021),. This system does not always meet the exact results for coarser step lengths. Every type of finite element system depends on the number of grid points.

Numerical and analytical techniques for solving conformable parabolic partial differential equations and conformable initial boundary value problems also have been investigated in Yavuz and Ozdemir (2017), Yavuz (2018) respectively. But the conformability transform is not only useful to solve local conformable fractional nonlinear dynamical systems of problems. Kocacoban et al. solved the Burgers–Fisher equation by applying various numerical schemes Kocacoban et al. (2011) that showed relatively faster convergence than other plots Lima et al. (2021). The collocation method is also a numerical method that the researcher used to obtain approximate solutions of non-linear parabolic types of partial differential equations in Hepson (2021). Hence, the researchers were developing the high-speed computers allows and improvements for the algorithm of several numerical methods to solve non-linear parabolic types of partial differential equations on both complex domain and complicated boundary conditions in different applications. For instance, this powerful series approach was applied by several researchers to find the solution of the Burger–Fisher equation, which is called a non-linear Parabolic Partial differential equation, see Behzadi and Araghi (2011).

However, each class of methods offers numerous and, in many ways, complementary benefits. Considering previous studies, it has been perceived that either analytical or numerical solutions of non-linear parabolic equations are very scarce. In the ideal case, we seek a method defined on arbitrary geometries that behaves regularly in any dimension and avoids the cost of time-consuming and mesh generation. As a result, many investigators have decided to advance too accurate and efficient numerical methods. Among those numerical methods, the Finite Difference scheme produces potential outcomes and has been widely used despite some limitations, such as being unable to obtain the solutions at every single point between two grid points. Another drawback is the computational cost to obtain higher accuracy of the numerical solution. Motivated by all the above studies, we come up with the idea to study the non-linear parabolic partial differential equations in one-dimensional space. Therefore, the main goal of this paper is to apply Guess-seidel iterative method to approximate the solution of the Non-linear parabolic partial differential equation and searching the results with its accuracy increases through iterative steps. The convergence of the present numerical scheme has been measured in the sense of average absolute error (AAE), maximum point-wise absolute error (L_∞), and root mean square error (L_2). The stability and confluence of the present techniques are also delved by using Von Neumann stability analysis techniques. In this paper, we consider the non-linear parabolic partial differential equation given by:

$$U_t = U_{xx} + G(x,t,U), \quad (a,b) \times (0,T) \qquad \textbf{Equation 1}$$

Subjected to both initial and boundary conditions are given by:

$$U(x,0) = U_0(x), a \le x \le b$$

$$U(a,t) = U_a(t), U(b,t) = U_b(t),\ 0 \le t \le T \qquad \textbf{Equation 2}$$

where α is arbitrary constant and $G(U)$, $U_0(x)$, $U_a(t)$, & $U_b(t)$ are smooth function in $[a,b] \times [0,T]$. This smoothness of this function is used for the existence of solutions in the domain. Moreover, the existence of solution of non-linear partial differential equation in the solution domain is studded in the references Ngoc et al. (2010), Truong et al. (2009), Ahmed (2017). To find the solution in this paper, the rectangular domain can be partitioned into sub-intervals given by:

$$a = x_0 < x_1 < x_2 < \cdots < x_{Mx} = b,\ 0 = t_0 < t_1 < t_2 < \cdots < t_{Nt} = T \qquad \textbf{Equation 3}$$

where $x_{j+1} = x_j + jh$, & $t_{n+1} = t_n + n\Delta t$, where $j = 0(1)Mx$ & $n = 0(1)N$. Mx & Nt are the maximum numbers of grid points respectively in the x and t direction. Therefore, this paper is organized as follows. Section two is a description of the numerical scheme, section three is confluence analysis, and section four is the results of numerical experiments. Section five is Discussions of numerical experiments; section six is the conclusion.

2. DESCRIPTION OF NUMERICAL SCHEME

Recall that non-linear parabolic partial differential equation in Equation 1 with their initial and boundary condition in Equation 2 and we want to approximate solution in the rectangular domain $[a,b] \times [0,T]$. Now to approximate this model problem, first, we want to discretize its derivative concerning in both temporal variable t and spatial variable x by using Taylor series expiation.

2.1. DISCRETIZATION OF TEMPORAL DERIVATIVE

Assume that $U(x,t)$ has continuous higher order partial derivative on its domain. Now, let us consider that $U(x,t_n) = U_n$, $\frac{\partial^p U(x,\ t_n)}{\partial t^p} = \partial_t^p U_n$ where $p \ge 1$ which we call, p^{th} order partial derivative of U concerning spatial variable x. Now the Taylor series expansions of $U_{n+1}, U_{n-1}, U_{n+2}$ and $U_{j\,n-2}$ about (x,t_n) given by

$$U_{n+1} = U_n + \Delta t \partial_t U_n + \frac{\Delta t^2}{2!}\partial_t^2 U_n + \frac{\Delta t^3}{3!}\partial_t^3 U_n + \frac{\Delta t^4}{4!}\partial_t^4 U_n + \cdots \qquad \textbf{Equation 4}$$

$$U_{n-1} = u_n - \Delta t \partial_t U_n + \frac{\Delta t^2}{2!}\partial_t^2 U_n - \frac{\Delta t^3}{3!}\partial_t^3 U_n + \frac{h^4}{4!}\partial_t^4 U_n + \cdots \qquad \textbf{Equation 5}$$

$$U_{n+2} = U_n + 2\Delta t \partial_t U_n + \frac{4\Delta t^2}{2!}\partial_t^2 U_n + \frac{8\Delta t^3}{3!}\partial_t^3 U_n + \frac{16\Delta t^4}{4!}\partial_t^4 U_n + \cdots \quad \textbf{Equation 6}$$

$$U_{n-2} = U_n - 2\Delta t \partial_t U_n + \frac{4\Delta t^2}{2!}\partial_t^2 U_n - \frac{8\Delta t^3}{3!}\partial_t^3 U_n + \frac{16\Delta t^4}{4!}\partial_t^4 U_n + \cdots \quad \textbf{Equation 7}$$

The first and second-order finite difference scheme for first, second, and third-order partial derivative concerning with temporal variable is

$$\partial_t U = \frac{U_{n+1}-U_n}{\Delta t} - \frac{\Delta t}{2}\partial_t^2 U_n \,,\ \partial_{tt} U = \frac{U_{n+1}-2U_n+U_{n-1}}{\Delta t^2} - \frac{\Delta t^2}{12}\partial_t^4 U_n$$

$$\partial_{ttt} U = \frac{U_{n+2}-2U_{n+1}+2U_{n-1}-U_{n-2}}{2\Delta t^3} - \frac{4\Delta t^2}{15}\partial_t^5 U_n \quad \textbf{Equation 8}$$

Now combining Equation 4, Equation 5, and Equation 7 we obtain

$$U_{n+1} + U_{n-1} + U_{n-2} = 3U_n - 2\Delta t \partial_t U_n + 3\Delta t^2 \partial_t^2 U_n - \frac{4\Delta t^3}{3}\partial_t^3 U_n + \frac{18\Delta t^4}{24}\partial_t^4 U_n + \cdots$$

Using Equation 8 in this difference in terms of second and third-order partial derivative we obtain

$$U_{n+1} + U_{n-1} + U_{n-2} = 3U_n - 2\Delta t \partial_t U_n + 3\Delta t^2\left[\frac{U_{n+1}-2U_n+U_{n-1}}{\Delta t^2} - \frac{\Delta t^2}{12}\partial_t^4 U_n\right] -$$

$$\frac{4\Delta t^3}{3}\left[\frac{U_{n+2}-2U_{n+1}+2U_{n-1}-U_{n-2}}{2\Delta t^3} - \frac{4\Delta t^2}{15}\partial_t^5 U_n\right] + \frac{18\Delta t^4}{24}\partial_t^4 u_n + \cdots$$

This implies that:

$$\partial_t U_n = \frac{1}{6\Delta t}[-2U_{n+2} + 10U_{n+1} - 9U_n + 2U_{n-1} + U_{n-2}] + \frac{11\Delta t^3}{36}\partial_t^4 U_n$$

Simplifying the difference result, we obtain the third-order finite difference scheme for the first-order finite difference scheme of the form

$$\partial_t U_n = \frac{1}{6\Delta t}[-2U_{n+2} + 10U_{n+1} - 9U_n + 2U_{n-1} + U_{n-2}]_j + E_1 \quad \textbf{Equation 9}$$

where $\tau_1 = \dfrac{11\Delta t^3}{36}\partial_t^4 U_n$ is its maximum local truncation error term?

2.2. DISCRETIZATION OF SPATIAL DERIVATIVE

Whit-out losing generality, the discretization of temporally derivative by using Taylor series expiation given by:

$$U_{j+1} = U_j + ?\partial_x U_j + \frac{h^2}{2!}\partial_x^2 U_j + \frac{h^3}{3!}\partial_x^3 U_j + \frac{h^4}{4!}\partial_x^4 U_j + \cdots \quad \textbf{Equation 10}$$

$$U_{j-1} = U_j - h\partial_x U_j + \frac{h^2}{2!}\partial_x^2 U_j - \frac{h^3}{3!}\partial_x^3 U_j + \frac{h^4}{4!}\partial_x^4 U_j + \cdots \quad \textbf{Equation 11}$$

$$U_{j+2} = U_j + 2h\partial_x U_j + \frac{4h^2}{2!}\partial_x^2 U_j + \frac{8h^3}{3!}\partial_x^3 U_j + \frac{16h^4}{4!}\partial_x^4 U_j + \cdots \quad \textbf{Equation 12}$$

$$U_{j-2} = U_j - 2h\partial_x U_j + \frac{4h^2}{2!}\partial_x^2 U_j - \frac{8h^3}{3!}\partial_x^3 U_j + \frac{16h^4}{4!}\partial_x^4 U_j + \cdots \quad \textbf{Equation 13}$$

Now using Equation 10, Equation 13, the second-order finite difference scheme for first and second-order partial derivative concerning spatial derivative is given by:

$$\partial_x U_j = \frac{U_{j+1} - U_{j-1}}{2h} - \frac{h^2}{6}\partial_x^3 U_j - \frac{h^2}{120}\partial_x^5 U_j - \cdots \quad \textbf{Equation 14}$$

$$\partial_x^2 U_j = \frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} - \frac{h^2}{12}\partial_x^4 u_j - \frac{h^2}{720}\partial_x^6 u_j - \cdots \quad \textbf{Equation 15}$$

$$\partial_x^3 U_j = \frac{U_{j+1} - 2U_{j+1} + 2U_{j-1} + U_{j-1}}{2h^3} - \frac{4h^2}{15}\partial_x^2 u_j - \cdots \quad \textbf{Equation 16}$$

$$\partial_x^4 U_j = \frac{U_{j+2} - 4U_{j+1} + 6U_j - 4U_{j-1} - U_{j-2}}{h^4} - \frac{13h^2}{72}\partial_x^6 u_j - \cdots \quad \textbf{Equation 17}$$

Without losing generality, combining Eqs. Equation 10, Equation 11, and Equation 13, we obtain the differential equation given by:

$$U_{j+1} + U_{j-1} + U_{j-2} = 3U_j - 2h\partial_x U_j + \frac{6h^2}{2}\partial_x^2 U_j - \frac{8h^3}{6}\partial_x^3 U_j + \frac{34h^4}{24}\partial_x^4 U_j - \frac{32h^5}{120}\partial_x^5 U_j + \frac{130}{720}\partial_x^6 U_j + \cdots$$

Substituting Equation 14 into this difference equation, we obtain:

$$-3h^2\partial_x^2 U_j = \frac{1}{4}U_{j+2} - 4U_{j+1} + \frac{15}{2}U_j - 4U_{j-1} + \frac{1}{4}U_{j-2} + \frac{2h^5}{15}\partial_x^5 U_j$$

Now multiplying both sides $-\frac{1}{3h^2}$ and simplifying the result, we obtain:

$$\partial_x^2 U_j = \frac{1}{12h^2}\left[-U_{j+2} + 16U_{j+1} - 30U_j + 16U_{j-1} - U_{j-2}\right]_n - \frac{2h^3}{45}\partial_x^5 U_j$$

By truncating the last (truncation) error terms from this difference scheme, we obtain a third-order central difference scheme for the second-order partial derivative of the model problem is given by:

$$\partial_x^2 U_j = \frac{1}{12h^2}\left[-U_{j+2} + 16U_{j+1} - 30U_j + 16U_{j-1} - U_{j-2}\right]_n + E_2 \quad \textbf{Equation 18}$$

where $\tau_2 = -\frac{2h^3}{45}\partial_x^5 U_j$ is their maximum local truncation error term?

Now substituting Equation 9, Equation 18 into Equation 1 we obtain the difference scheme:

$$\frac{1}{6\Delta t}\begin{bmatrix} -2U_{jn+2} + 10U_{jn+1} - \\ 9U_{jn} + 2U_{jn-1} + U_{jn-2} \end{bmatrix} = \frac{1}{12h^2}\begin{bmatrix} -U_{j+2n} + 16U_{j+1n} - \\ 30U_{jn} + 16U_{j-1n} - U_{j-2n} \end{bmatrix} + G(x_j, t_n, U_{jn})$$

This implies that:

$$4U_{jn+2} - 20U_{jn+1} + 18U_{jn} - 4U_{jn-1} - 2U_{jn-2} = \mu\begin{bmatrix} U_{j+2n} - \\ 16U_{j+1n} + 30U_{jn} - 16U_{j-1n} + U_{j-2n} \end{bmatrix} - 12\Delta t G(x_j, t_n, U_{jn})$$

where $\mu = \frac{\Delta t}{h^2}$. Then we can rewrite the iterative scheme by using the Gauss-seidel iterative scheme as a form of:

$$\alpha U_{j\,n}^{(k+1)} - 4U_{j\,n-1}^{(k+1)} - 2U_{j\,n-2}^{(k+1)} = -4U_{j\,n+2}^{(k)} + 20U_{j\,n+1}^{(k)} + \mu\big(U_{j+2\,n}^{(k)} - 16U_{j+1\,n}^{(k)} - 16U_{j-1\,n}^{(k)} + U_{j-2\,n}^{(k)}\big) + 12\Delta t G(x_j, t_n, U_{jn}^{(k)})$$

where $\alpha = 18 - 30\mu$ and $k = 1,2,3, \ldots$. From this scheme, we can rewrite this, into the embedded matrix form of Gauss-seidel iterative scheme:

$$(L + D)U_j^{(k+1)} = \widehat{U} U_j^{(k)} + b_j^{(k)} \quad \textbf{Equation 19}$$

Where L , D and $\widehat{U}$ are lower, diagonal, and an upper triangular matrix is respectively given by:

$$L = \begin{bmatrix} 0 & 0 & 0 & & 0 & 0 \\ -4 & 0 & 0 & \cdots & 0 & 0 \\ -2 & -4 & 0 & & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & -2 & 0 \end{bmatrix}, D = \begin{bmatrix} \alpha & 0 & 0 & & 0 & 0 \\ 0 & \alpha & 0 & \cdots & 0 & 0 \\ -0 & 0 & \alpha & & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \alpha \end{bmatrix} \text{ and } \widehat{U} = \begin{bmatrix} 0 & 20 & -4 & & 0 & 0 \\ 0 & 0 & 20 & \cdots & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

And

$$b_j^{(k)} = \mu\left(U_{j+2\,n}^{(k)} - 16U_{j+1\,n}^{(k)} - 16U_{j-1\,n}^{(k)} + U_{j-2\,n}^{(k)}\right) + 12\Delta t G\left(x_j, t_n, U_{jn}^{(k)}\right) - 4U_{j\,Nt+1}^{(k)} + 20U_{j\,Nt+2}^{(k)} + 4U_{j\,,0}^{(k+1)} + 2U_{j,-1}^{(k+1)} + 2U_{j,0}^{(k+1)} \quad \textbf{Equation 20}$$

Where $\tau_{jn} = \left[\frac{2h^3}{45}\partial_x^5 - \frac{11\Delta t^3}{36}\partial_t^4\right] U_{jn}$ is a local truncation error? Hence, the simplified form of Guess-seidel iterative scheme is:

$$U_j^{(k+1)} = AU_j^{(k)} + B_j \quad \textbf{Equation 21}$$

where $A = (L + D)^{-1}\hat{U}$ and $B_j = (L + D)^{-1}b_j^{(k)}$

Thus, using Guess-seidel iterative scheme in Equation 18 by rewriting MATLAB program, we obtain the solution of model problem and validity of proposed numerical scheme.

3. CONSISTENCY AND CONVERGENCE ANALYSIS

3.1. THE CONSISTENCY OF THE PROPOSED SCHEME

Since from the general iterative scheme, local truncation error is $\tau_{jn} = \left[\frac{2h^3}{45}\partial_x^5 - \frac{11\Delta t^3}{36}\partial_t^4\right]U_{jn}$. Hence by using the definition referenced in Morton and Mayers (2005), we have:

$$\lim_{h,\Delta t\to 0}\left|\frac{2\alpha h^3}{45}\partial_x^5 - \frac{11\Delta t^3}{36}\partial_t^4\right| = \lim_{Mx,Nt\to\infty}\left|\tau_{jn}\right| \cong 0$$

Hence this indicated that, as simultaneously both step-length and time step approach to zero (i.e $h, \Delta t \to 0$), the truncation error in difference scheme is approximate to zero (i.e.,$h, \Delta t \to 0, \tau_{jn} \to 0$). So, this shows that, the above iterative scheme is consistent.

3.2. CONVERGENCE ANALYSIS

The convergence of the proposed numerical method is investigated by using matrix form convergence analysis. Such an approach has been used in many textbooks and different recent articles. As it worked in reference Mohanty and Jha (2005), JAIN et al. (1984) assume that $U_j = [U_1, U_1, \dots, U_{Mx}]^t$ is the exact solution of the problem in Equation 1 and it can be a writer as:

$$U_j = AU_j + B_j \qquad \textbf{Equation 22}$$

Subtraction Eq. Equation 21 form Eq. Equation 22 and substituting $\in^{(k)} = U - U_j^k$, we obtain:

$$\in^{(k+1)} = A^k \in^{(k)} \qquad \textbf{Equation 23}$$

where $k = 1,2,3, \dots$ and it follows that $\in^{(k)} = A^k \in^{(0)}$. $\in = [\in_1, \in_2, \in_3, \dots, \in_{Mx}]^t$. Since in our work we follow that error produced in this scheme is less-than principal local truncation error produced in sequences of the scheme. It means that $\left|\in_j^{(k)}\right| \ll \tau_j(h, \Delta t)$ where is maximum local truncation error term at j^{th} point. This shows the stability of the scheme. Now assuming that matrix A is irreducible and monotone Mohanty and Jha (2005),. This shows that the inverse of A exists, and its elements are nonnegative. Hence, using matrix norm, from Equation 23, we get

$$\left\|\in_j\right\| \ll \|\boldsymbol{A}\|.\|\tau(h, \Delta t)\|$$

Thus, we define the norms of the matrix as constant which is given by $K = \|\boldsymbol{A}\| = \max_{j} \sum_{j=1}^{Mx} |a_{lj}|$

where $A = \{a_{lj}\}$ and norm of truncation error $\|\tau(h, \Delta t)\| = \max_{j} |\tau_j(h, \Delta t)|$. Therefore, we have

$$\|\epsilon_j\| \ll K \max_{j} |\tau_j(h, \Delta t)|$$

However, all error in the scheme is bounded. Thus, we have $\|\in\| \ll O(h^3 + \Delta t^3)$.

Theorem 1: Let A in Equation 23 be a square matrix and λ_j is distinct Eigenvalue of matrix A. Then $\lim_{k\to\infty} A^k = 0$ if $\|A\| < 1$ or $\rho(A) < 1$ where $\rho(A)$ is the spectral radius of matrix A and $\rho(A) = \max_{j}(\lambda_j)$.

Proof: If $\|A\| < 1$, we have $\|A^k\| \leq \|A\|^k$ and $\left\|\lim_{k\to\infty} A^k\right\| \leq \left\|\lim_{k\to\infty} A\right\|^k = 0$. For simplicity, assume that all the eigenvalues of A are distinct. Then, there exists a similarity transformation P, such that $A = PDP^{-1}$ where D is the diagonal matrix having the eigenvalues of A on the diagonal. Therefore, $A^k = PD^kP^{-1}$ and

$$D^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & & \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{Mx}^k \end{bmatrix}$$

This is implying $\lim_{k\to\infty} A^k = 0$ if and only if all Eigenvalue of A satisfies $|\lambda_j^k| < 1$. Therefore $\rho(A) < 1$.

Theorem 2: A necessary and sufficient condition for convergence of an iterative method of the form given in Equation 21 is that the eigenvalues of the iteration matrix satisfy$|\lambda_j A| < 1, j = 1,2, \dots Mx$.

Proof: Prove of this theorem is given in JAIN et al. (1984).

Therefore, by supporting these two theorems with the above theoretical and numerical error bound, the present method is convergent with third-order convergence. Hence to measure the accuracy of the proposed method, we use norms of average absolute error (AAE), root mean square (RMS) error (L_2) and maximum point-wise absolute error (L_∞). These norms are calculated as follows:

$$L_2 = \sqrt{\frac{1}{Mx}\sum_{j=1}^{Mx}\left|U(x_j,t) - U_{j\,Nt}\right|^2}\ , L_\infty = \max_{1\leq j\leq Mx}\left|U(x_j,t) - U_{j\ Nt}\right|, \text{AAE} = \frac{1}{Mx}\sum_{j=1}^{Mx}\left|U(x_j,t) - U_{j\,Nx}\right|,$$

Where $U(x,t)$ and $U_{j\,n}$ are the respectively exact and numerical solutions of the given model example.

4. RESULTS OF NUMERICAL EXPERIMENTS

To demonstrate the applicability of the methods, three model examples have been considered and they are bellowed.

Example 1. Consider the following nonlinear parabolic problem:

$$U_t = \alpha U_{xx} - U + U^2 + f(x,t);\ 0 < x < 1\ ,\ 0 < t < T$$

Initial condition $U(x,0) = sin(\pi x)$,

Boundary condition $U(0,t) = U(1,t) = 0$

Exact Solution: $u(x,t) = exp(-t)sin(\pi x)$.

Here source function: $f(x,t) = exp(-t)sin(\pi x)(\pi\text{^}2 - exp(-t)sin(\pi x))$

Example 2. Consider the following nonlinear parabolic problem:

$$U_t = U_{xx} - H(U - \gamma) + f(x,t);\ 0 < x < 1\ ,\ 0 < t < T$$

Initial condition $U(x,0) = sin(\pi x)$,

Dirichlet boundary condition $U_x(0,t) = U_x(1,t) = 0$

Exact Solution: $u(x,t) = exp(-\pi^2 t)sin(\pi x)$.

Here source function: $f(x,t) = \kappa(exp(-\pi^2 t)sin(\pi x) - \gamma\ )$

Example 3. Consider the following nonlinear parabolic problem:

$$U_t = U_{xx} - H(U - \gamma) + |U_x|^2 + f(x,t);\ 0 < x < 1\ ,\ 0 < t < T$$

Initial condition $U(x,0) = sin(\pi x)$,

Robin boundary condition $U_x(0,t) = U_x(1,t) = \pi exp(-\pi^2 t)$

Exact Solution: $u(x,t) = exp(-\pi^2 t)sin(\pi x)$.

Here source function: $f(x,t) = \kappa(exp(-\pi^2 t)sin(\pi x) - \gamma\ ) + |\pi exp(-\pi^2 t)\cos^2(\pi x)|$

Table 1 Displaying the efficiency of the proposed scheme by listing Exact solution, Numerical solution, and point-wise absolute error, for problem give an example one for Mx=20 and at time t=1.154 when computations domain carried out until final time T=2

Values of x	Numerical and Exact solution		Point-wise Absolute Error
x	U_j	$U(x_j,t)$	L_∞
0.05	$1.5428E-06$	$1.5226E-06$	$2.0255E-08$
0.25	$6.9737E-06$	$6.8822E-06$	$9.1556E-08$
0.5	$9.8624E-06$	$9.7329E-06$	$1.2948E-07$
0.8	$5.7970E-06$	$5.7208E-06$	$7.6106E-08$
0.95	$1.5428E-06$	$1.5226E-06$	$2.0255E-08$

Table 2 Displaying the efficiency of the proposed scheme by listing point average absolute error, Root Mean Square Error, and pointwise maximum absolute error for problem give an example one when computations domain carried out until final time T=1 for different mesh size h and time step Δt

Specified Mesh size		Estimated errors at Specified Mesh size		
h	Δt	L_∞	L_2	AAE
0.05	0.01	$1.2948E-07$	$2.8952E-08$	$6.4739E-09$
0.025	0.01	$1.1151E-07$	$1.7631E-08$	$2.7877E-09$
0.0125	0.008	$1.0666E-07$	$1.1925E-08$	$1.3332E-09$
0.01	0.008	$1.0597E-07$	$1.0597E-08$	$1.0597E-09$

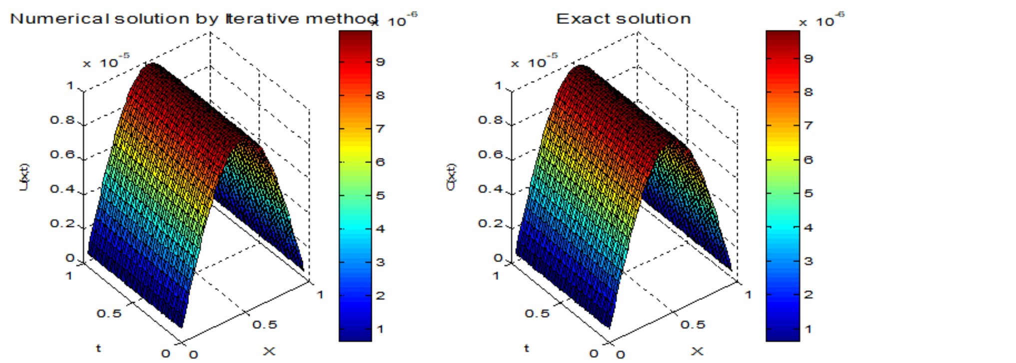

Figure 1 Solution profile for the solution of example one on the uniform mesh of maximum number grid point is $\boldsymbol{Mx} = \mathbf{50}$ & time step is $\Delta \boldsymbol{t} = \mathbf{0.01}$

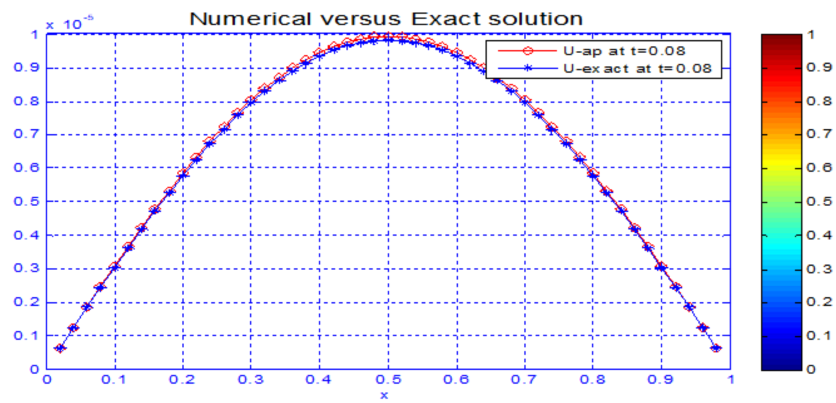

Figure 2 Graphical comparison of numerical versus exact solution of example one using a uniform mesh with maximum number grid point $\boldsymbol{Mx} = \mathbf{50}$ & time step is $\Delta \boldsymbol{t} = \mathbf{0.01}$

Table 3 Comparison of maximum point-wise absolute error and Root Mean Square error for problem give an example two for computations domain carried out until final time $T = 0.5$ with different mesh size h and time step $\Delta t = 0.01, \kappa = 0.25$ and $\gamma = 0.25$

Number of grid points	Estimated norm of error with Specified Mesh size			
	Result by R. Alharbi in Alharbi (2020)		Result by Present Methods	
Mx	L_∞	L_2	L_∞	L_2
32	$2.797E-03$	$2.359E-03$	2.22E-06	3.92E-07
64	$1.735E-03$	$5.897E-04$	2.22E-06	2.77E-07
128	$4.337E-04$	$1.474E-04$	2.22E-06	1.96E-07
256	$1.084\,E-04$	$3.685E-05$	2.22E-06	1.39E-07
512	$2.710E-05$	$9.213E-06$	2.22E-06	9.80E-08

Table 4 Comparison of maximum point-wise absolute error and Root Mean Square error for problem give an example two for computations domain carried out until final time $T = 0.5$ with different mesh size h and time step $\Delta t = 0.01, \kappa = 1$ and $\gamma = 0.5$

Number of grid points	Estimated norm of error with Specified Mesh size			
	Result by R. Alharbi in Alharbi (2020)		Result by Present Methods	
Mx	L_∞	L_2	L_∞	L_2
32	$2.123E-03$	$1.791E-03$	2.21E-05	3.91E-06
64	$5.310E-04$	$4.479E-04$	2.21E-05	2.76E-06

128	$1.327E-04$	$1.119E-04$	2.21E-05	1.95E-06
256	$3.319E-05$	$2.799E-05$	2.21E-05	1.38E-06
512	$8.298E-06$	$6.999E-06$	2.21E-05	9.77E-07

Table 5 Comparison of maximum point-wise absolute error and Root Mean Square error for problem give an example two in computations domain carried out until final time $T=0.5$ with different mesh size h and time step $\Delta t = h$, $\kappa = 1$ and $\gamma = 0.5$

Number of grid points	Estimated norm of error with Specified Mesh size			
	Result by R. Alharbi in Alharbi (2020)		**Result by Present Methods**	
Mx	L_∞	L_2	L_∞	L_2
32	$2.123E-03$	$1.791E-03$	6.71E-06	1.19E-06
64	$1.289E-03$	$1.087E-03$	1.39E-05	1.7356eE-06
128	$7.015E-04$	$5.917E-04$	2.86E-05	2.53E-06
256	$3.649E-04$	$3.078E-04$	5.77E-05	3.60E-06
512	$1.860E-04$	$1.569E-04$	1.16E-04	5.11E-06

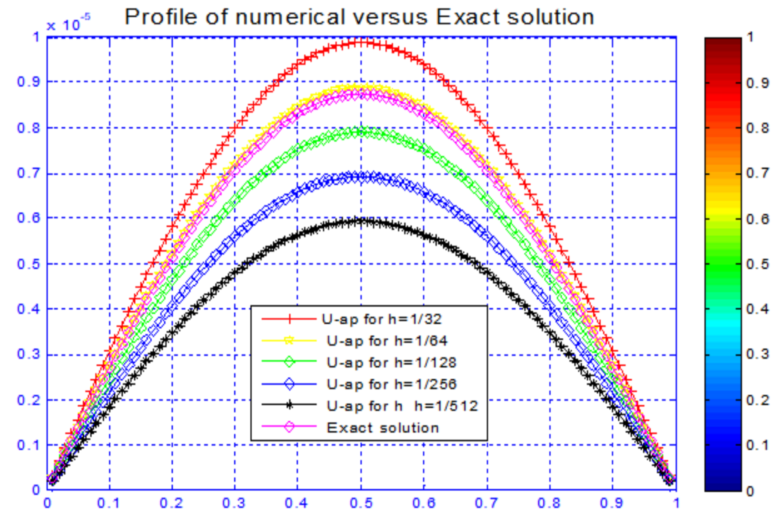

Figure 3 Graphical representation of numerical solution for example two using uniform mesh mesh-size $\boldsymbol{h = 0.05}$ & time step and $\boldsymbol{\Delta t = 0.01}$ and $\boldsymbol{\kappa = 0.25}$ and $\boldsymbol{\gamma = 0.25}$

Table 6 Comparison of maximum point-wise absolute error and Root Mean Square error for problem give an example three in computations domain carried out until final time $T=0.5$ with different mesh size h, $\kappa = 0.25$ and $\gamma = 0.5$

Number of grid points	Estimated norm of error with Specified Mesh size			
	Result by R. Alharbi in Alharbi (2020) time step$\Delta t = h^2$		Result by Present Methods time step$\Delta t = 0.0125$	
Mx	L_∞	L_2	L_∞	L_2
32	$1.613E-02$	$1.358E-02$	1.76E-05	3.12E-06
64	$4.020E-03$	$3.383E-03$	1.76E-05	2.20E-06
128	$1.00E-03$	$8.451E-04$	1.76E-05	1.56E-06
256	$2.509E-04$	$2.112E-04$	1.76E-05	1.10E-06
512	$6.274E-05$	$5.280E-05$	1.76E-05	7.79E-07

Table 7 Comparison of maximum point-wise absolute error and root mean square error for problem give an example three in computations domain carried out until final time $T = 0.5$ with different mesh size h, $\kappa = 1$ and $\gamma = 0.5$

Number of grid points	Estimated norm of error with Specified Mesh size			
	Result by R. Alharbi in Alharbi (2020) time step $\Delta t = h^2$		Result by Present Methods time step $\Delta t = 0.025$	
Mx	L_∞	L_2	L_∞	L_2
32	$5.491E-03$	$4.622E-03$	7.18E-07	1.27E-07
64	$1.372E-03$	$1.154E-03$	7.15E-07	8.94E-08
128	$3.429E-04$	$2.886E-04$	7.15E-07	6.32E-08
256	$8.573E-05$	$7.215E-05$	7.14E-07	4.47E-08
512	$2.143E-05$	$1.803E-05$	7.14E-07	3.16E-08

Table 8 Comparison of maximum point-wise absolute error and root mean square error for problem give an example three in computations domain carried out until final time $T = 0.5$ with the different mesh size of $h = \Delta t$, $\kappa = 1$ and $\gamma = 0.5$

Number of grid points	Estimated norm of error with Specified Mesh size			
	Result by R. Alharbi in Alharbi (2020)		Result by Present Methods	
Mx	L_∞	L_2	L_∞	L_2
32	$4.622E-03$	$5.491E-03$	5.76E-07	1.02E-07
64	$2.346E-03$	$2.788E-03$	1.09E-06	1.36E-07
128	$1.181E-03$	$1.404E-03$	2.13E-06	1.88E-07
256	$5.930E-04$	$7.047E-04$	4.18E-06	2.61E-07
512	$2.970E-04$	$3.529E-04$	8.27E-06	3.65E-07

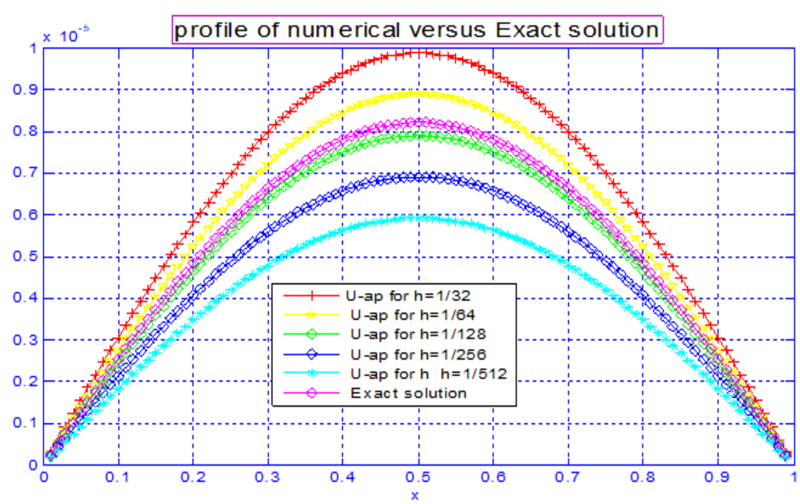

Figure 4 Graphical representation of the numerical solution of example three using a uniform mesh with different mesh-size & time step is $\Delta t = 0.01$, $\kappa = 1$ and $\gamma = 0.5$

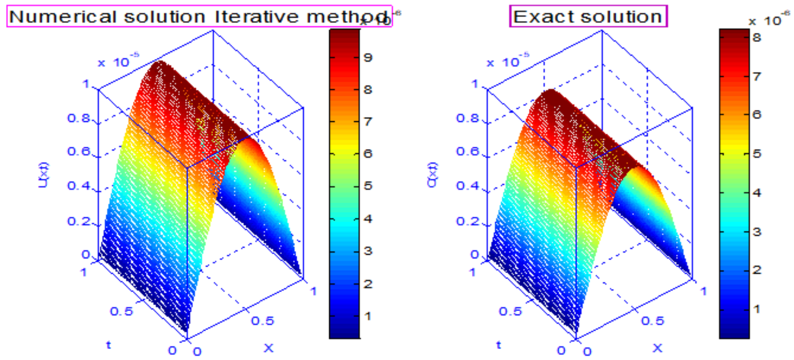

Figure 5 Solution profile for the solution of example three on the uniform mesh of maximum number grid point is $Mx = 100$, the time step is $\Delta t = 0.01$ $\kappa = 1$ and $\gamma = 0.5$

5. DISCUSSION

In this paper, the third-order iterative scheme is presented to solve a one-dimensional non-linear parabolic partial differential equation. To demonstrate computation for the accuracy of percent of the method with the pre-existing method, three model examples are solved by taking different values for step size h, and time step k. The computation of numerical results obtained by the present method has been presented in terms of average absolute error, root means square error and maximum point-wise absolute error. Results presented in Table 1 and Table 2 show that average absolute error (AAE), roots mean square error (L_2) and point-wise maximum absolute error norm (L_∞) decreases as mesh-size h and k are decreases. Again, as we can see the comparison of the numerical and exact solution of the model problem given in example one summarized in Table 1 at selected grid points shows that the numerical solution is in good agreement with its exact solution. Also, the results of example two, given in Table 3 up to Table 5 show that, the accuracy of the present iterative method increases, and it's superior to the accuracy of the scheme in Alharbi (2020). Figure 1,Figure 2, Figure 3,Figure 4, Figure 5 shows the physical background of the solution within the traditional form of future expected interest rates for each maturity. It means it shows that the forecasting rate of change of the total amount of interest rates is increasing to decreasing for each maturity of U in any sections of $0 < x < 1$ with a balance to the net inflow of interest across 0 to the 0.5-time interval. Also, in this case, the accuracy of the present method is rapidly increased and it's superior to the accuracy of the scheme in Alharbi (2020). Further, as shown in Figure 3, the proposed method approximates the solution very well. The result presented in Table 6,Table 7,Table 8 also shows that; the root mean square error norm and maximum absolute error norm are decreased uniformly for solving example three with different values of step-length h, time-step Δt, and constant f variation (K, γ). As we predict from this result, the accuracy of the present method is rapidly increased and it's superior to the accuracy of the scheme in Alharbi (2020). Therefore, the accuracy of the present method confirmed the established numerical error bound. Hence in solving all applied three model examples, the results given in tables in terms of error norm and graphs of numerical versus exact solutions are further confirmed that the computational rate of convergence and theoretical estimates error bounds

6. CONCLUSION

A new approach, a third-order iterative scheme is used to solve nonlinear parabolic partial differential equations numerically and the result is presented in table and graph. The comparison of the accuracy indicates that; the present method is the more convenient, reliable, and effective scheme. As it can be seen from the table and graphs, the present methods, improve the accuracy, by minimizing the number of grid points in a time interval and an equal number of grid points in the spatial interval with pre-existing methods. This shows that the present method avoids the cost of time-consuming and meshes generation. In a summary, the third iterative scheme is capable to solve nonlinear parabolic partial differential equations. Based on the findings, this method is well approximate and gives better accuracy for the numerical solution with a decreasing step size h, and fixed time step Δt.

AUTHOR CONTRIBUTIONS

The author planned to work by this scheme, initiated the Research idea, suggested their experiment, conducted the experiments, and analyzed the empirical results. Also, the Author developed the mathematical modeling and examined the numerical validation and was written their manuscript properly. The author reviews its results and approves the final version of the manuscript.

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NOMENCLATURE

L_2	Root mean square error	h	step length
L_∞	Maximum Pointwise error	Δt	time step
AAE	Average Absolute Error	Mx,	maximum number grid point in the spatial direction,
Nt	Maximum Number of the grid point in temporal direction		

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