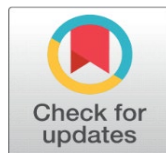


## A STUDY ON ALPHA POWER LOMAX DISTRIBUTION

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### ABSTRACT

In this paper, we refer to the new distribution an alpha power Lomax distribution. Various properties of the proposed distribution are obtained including mode, quantiles, entropies, and order statistics are obtained. Parameters of the proposed distribution are estimated using maximum likelihood, ordinary least squares and weighted least squares. Simulation study is conducted to compare between estimators.

**Keywords:** Alpha Power Lomax Distribution, Maximum Likelihood Estimation, Renyi Entropies, Stress–Strength Reliability and Order Statistics

## 1. INTRODUCTION

Improvement over standard distributions has gained popularity in statistical theory over the past few years. Typically, new distributions are created by combining existing distributions or adding a new parameter using generators. Alpha power transformation (APT) a novel approach for integrating an additional parameter in continuous distribution, was introduced by [Mahdavi and Kundu \(2017\)](#). Basically, the concept was put forth to add skewness to the baseline distribution. Then: Let  $g(x)$  be the probability density function (PDF) of a continuous random variable  $X$ :

$$f(x) = \begin{cases} \frac{\text{Log}a}{a-1} a^{G(x)} g(X) & \text{if } a \geq 0, a \neq 1 \\ G(x) & \text{if } a = 1 \end{cases}$$

Equation 1

The corresponding cumulative distribution function (CDF)

$$F(x) = \begin{cases} \frac{\alpha G(x)-1}{\alpha-1} & \text{if } \alpha \neq 1 \\ G(x) & \text{if } \alpha = 1 \end{cases}$$

Equation 2

Where  $\alpha$  is the shape parameter and  $G(x)$  and  $g(x)$  are the CDF and the PDF of the baseline distribution.

Mahdavi and Kundu (2017) developed a brand-new method for creating distributions by adding a second shape parameter to a well-known baseline distribution. They used the exponential baseline distribution to introduce the alpha power exponential distribution, and they researched the fundamental characteristics as well as parameter estimation for the suggested distribution. Alpha power Weibull distribution (APWD) was introduced by Nassar et al. (2017). The capacity of APWD to simulate monotone and non-monotone failure rate functions, which are often used in reliability studies, is what gives its significance. Moments, quantiles, entropy, order statistics, mean residual life function, and stress-strength were among the attributes of APWD that were discovered. The parameters were estimated using the maximum likelihood method. The significance of the APWD was demonstrated using of actual data sets. Devi et al. (2017) introduced the entropy of the Lomax probability distribution. They derived entropy of a Lomax probability distribution as well as their order statistics which is used in business, economics, actuarial modelling, waiting problems, and biological sciences.

This study's objective is to provide a new more adaptable distribution. The structure of this article is as follows: We go over the distribution of Alpha power Lomax (APL) in Section 2. In Section 3, new statistical properties are investigated. The Renyi entropy is derived in Section 4. In Section 5, the stress-strength reliability is described. Section 6 presents the order statistics from the Alpha power Lomax. Section 7 examines estimate techniques. Section 8 provides a simulation study. In Section 9, application is offered.

## 2. ALPHA POWER LOMAX DISTRIBUTION

In this section, we will define a new model is called alpha power Lomax distribution, denoted (APL), with three parameters  $(\alpha, \lambda, \beta)$ .

According to El-Houssainy et al. (2016) cumulative's distribution function of Lomax distribution with scale parameter  $\lambda > 0$  is

$$G(x) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-\beta}, \quad x > 0$$

Equation 3

where,  $\beta > 0$  and  $\lambda > 0$ .

The probability density function corresponding to (3) reduces to

$$g(x) = \frac{\beta}{\lambda} \left(1 + \frac{x}{\lambda}\right)^{-(\beta+1)}, \quad x > 0$$

Equation 4

The CDF of APL is generated by renaming Equation 3 as a baseline cumulative function in Equation 2.

$$F(x) = \begin{cases} \frac{1 - [a^{1 - (1 + \frac{x}{\lambda})^{-\beta}} - 1]}{\alpha - 1} & \text{if } a \neq 1 \\ \frac{x^{-\beta}}{1 - (1 + \frac{x}{\lambda})^{-\beta}} & \text{if } a = 1 \end{cases} \tag{Equation 5}$$

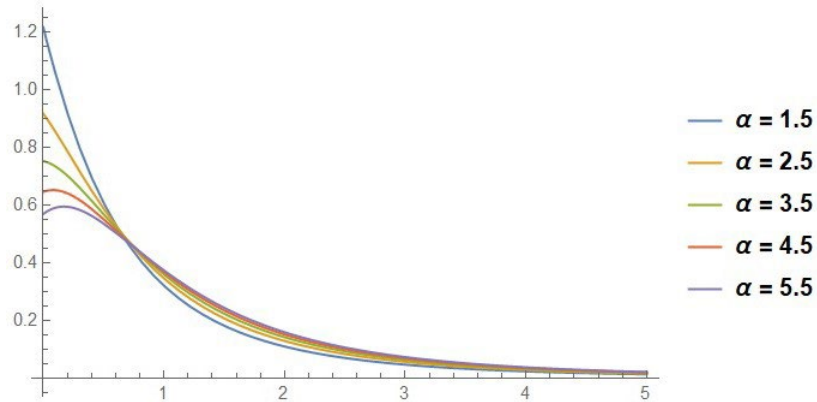
and corresponding PDF of APL is defined as follows

$$f(x) = \begin{cases} \frac{\log a}{\alpha - 1} a^{1 - (1 + \frac{x}{\lambda})^{-\beta}} \beta (1 + \frac{x}{\lambda})^{-(\beta+1)} & \text{if } a \neq 1 \\ \frac{\beta}{\lambda} (1 + \frac{x}{\lambda})^{-(\beta+1)} & \text{if } a = 1 \end{cases} \tag{Equation 6}$$

where  $\lambda$  is the scale parameter and  $\alpha$  is the shape parameter.

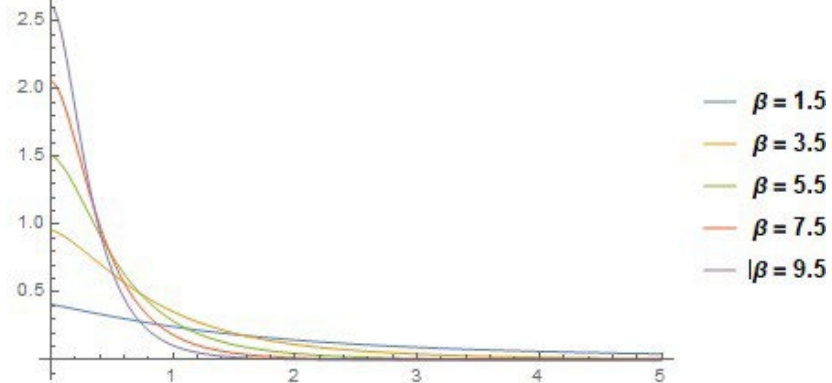
The PDF curves for the APL distribution are shown in Figure 1, Figure 2, Figure 3 with various parameter values.

**Figure 1**



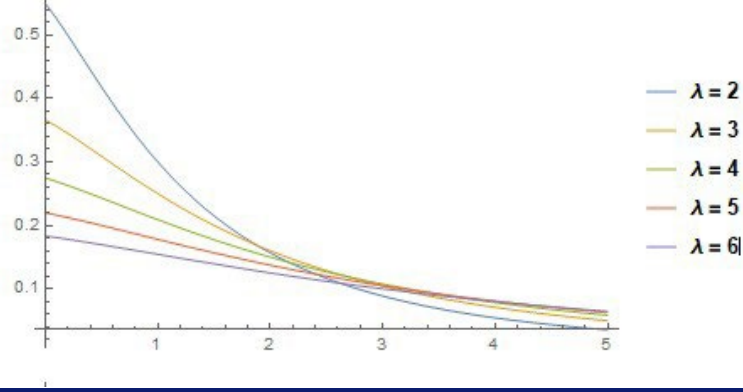
**Figure 1** The APL's PDF curves with  $\lambda = 2, \beta = 3$

**Figure 2**



**Figure 2** The APL's PDF curves with  $\alpha = 3, \lambda = 2$

**Figure 3**



**Figure 3** The APL's PDF curves with  $\alpha = 3, \beta = 2$

Figure 1, Figure 2, Figure 3 show that for various values of the parameters, the APLD density curve is uni-modal and positively skewed.

### 2.1. THE SURVIVAL AND THE HAZARD FUNCTIONS

The following forms could be used to represent the survival and hazard rate functions for the APL distribution, respectively.

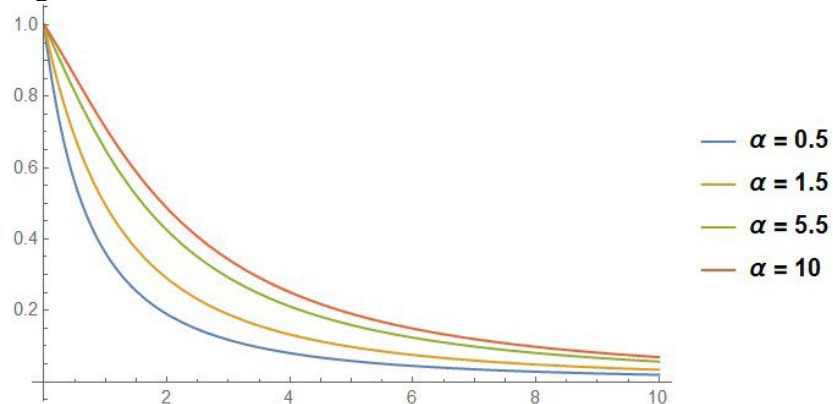
$$S(x) = \frac{a}{a-1} [1 - a^{-(1+\frac{x}{\lambda})^\beta}], \quad x > 0. \tag{Equation 7}$$

and

$$h(x) = \frac{\beta \log[a] a^{-(1+\frac{x}{\lambda})^\beta} (1+\frac{x}{\lambda})^{-(\beta+1)}}{\lambda (1 - a^{-(1+\frac{x}{\lambda})^\beta})}, \quad x > 0. \tag{Equation 8}$$

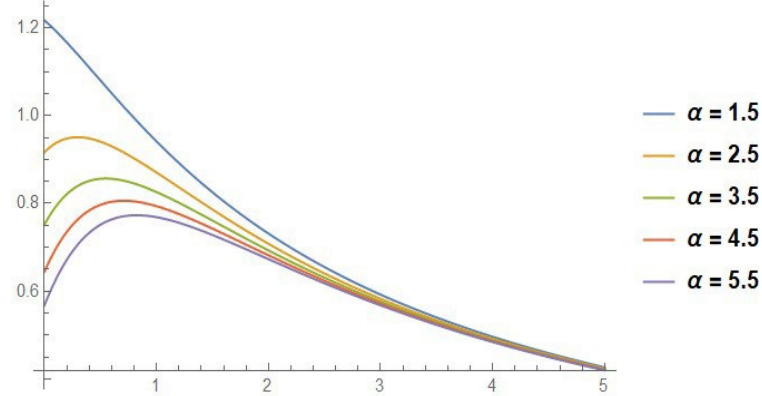
Figure 4, Figure 5 display the survival function of APLD and the hazard function for different values of parameters.

**Figure 4**



**Figure 4** The APL's survival function curves with  $\lambda = 2, \beta = 2$

**Figure 5**



**Figure 5** The The APL's hazard function curves show with  $\lambda = 2$  and  $\beta = 3$ .

### 3. STATISTICAL PROPERTIES

#### 3.1. QUANTILE FUNCTION

Let  $u = F(x)$  where  $U$  follows uniform  $(0, 1)$ . By using the transformation method, we consider the random variable  $X$  of APL as follows

$$Q(u) = x_u = \lambda \left[ \left( \frac{\log(\alpha / (u(\alpha - 1) + 1))}{\log \alpha} \right)^{-\frac{1}{\beta}} - 1 \right].$$

Equation 9

##### 3.1.1. THE MEDIAN

The median of the APL distribution can be obtained by putting  $u = 0.5$  in Equation 9 as follows

$$Med = \lambda \left[ \left( \frac{\log(2\alpha / (\alpha + 1))}{\log \alpha} \right)^{-\frac{1}{\beta}} - 1 \right].$$

Equation 10

#### 3.2. THE MODE

The mode of the distribution can be found by solving the following equation

$$\frac{\partial f}{\partial x} = 0$$

By taking the derivative of Equation 6 and equating it to zero and solving for  $x$ , mode becomes

$$x = \lambda \left[ \left( \frac{\beta + 1}{\beta \log \alpha} \right)^{-\frac{1}{\beta}} - 1 \right]$$

Equation 11

This equation is not linear. The Newton-Raphson method can be used to solve it numerically.

### 3.2.1. SKEWNESS AND KURTOSIS

Because the APL's moments are not exist, we'll apply an alternate forms using quartiles. Bowley skewness was considered as an alternate measure to determine asymmetry (Kenney and Keeping (1962)) of a distribution that takes the form

$$R = \frac{[Q(\frac{3}{4}) - 2Q(\frac{1}{2}) + Q(\frac{1}{4})]}{|Q(\frac{3}{4}) - Q(\frac{1}{4})|}$$

where Q represents the quantile function from Equation 9. Moors (1988) proposed a different method to determine the kurtosis of the distribution based on octiles, and it has the following form

$$M = \frac{[Q(\frac{7}{8}) - Q(\frac{5}{8})] + [Q(\frac{3}{8}) - Q(\frac{1}{8})]}{|Q(\frac{6}{8}) - Q(\frac{2}{8})|}$$

**Table 1**

**Table 1 The Mode, Median, Skewness and Kurtosis of APL Distribution for Different Values of the Parameters.**

$\alpha$	$\lambda$	$\beta$	Mode	Median	Skewness	Kurtosis
4.5	0.5	2	0.000679	0.373801	0.341431	1.58388
8.5	0.5	2	0.097225	0.458852	0.327832	1.57484
12.5	0.5	2	0.148810	0.512296	0.321915	1.57231
4.5	1	2	0.001358	1.024560	0.341431	1.58388
4.5	3	2	0.004074	3.07377-	0.341431	1.58388
4.5	6	2	0.008149	6.147550	0.341431	1.58388
4.5	0.5	2.5	0.014549	0.281491	0.309599	1.50100
4.5	0.5	3	0.020491	0.225435	0.288054	1.45078
4.5	0.5	3.5	0.022919	0.187873	0.272518	1.41727

Table 1 the following:

- 1) For constant values of  $\lambda$  and  $\beta$ , the median, mode increasing as  $\alpha$  increases, while the kurtosis and skewness decrease.
- 2) For constant values of  $\alpha$  and  $\beta$ , the median and mode increase while  $\lambda$  is increasing but the kurtosis and skewness remain stable.
- 3) For constant values of  $\lambda$  and  $\alpha$ , the mode increasing as  $\beta$  increases, while the median, kurtosis, and skewness decrease.

### 4. RENYI ENTROPY OF APL

Renyi entropy was invented by Renyi (1959). The variation of the uncertainty is measured by a random variable's entropy X. Additionally, Song (2001) has effectively used the theory of entropy to a variety of applications, including information theory, engineering, and physics.

Theorem 1: The Renyi entropy of APL

$$R = \frac{\rho}{1-\rho} \log\left(\frac{\alpha \log \alpha}{\alpha-1}\right) + \frac{1}{1-\rho} \log\left[\frac{\beta^\rho \lambda}{\lambda^\rho \beta} \sum_{k=0}^{\infty} \frac{(-\log \alpha)^k}{k!} \rho^k \frac{\beta}{\beta k + \beta \rho + \rho - 1}\right]$$

Equation 12

Proof. For the density function f(x), the Renyi entropy is defined as:

$$K = \frac{1}{1-\rho} \log \int_0^{+\infty} f(x)^\rho dx, \quad \rho \neq 1$$

Equation 13

Using Equation 6, we obtain

$$= \frac{1}{1-\rho} \log\left[\left(\frac{\alpha \log \alpha}{\alpha-1}\right)^\rho \frac{\beta^\rho}{\lambda^\rho} \int_0^{+\infty} \left(\alpha^{-(1+\frac{x}{\lambda})^\beta} \left(1 + \frac{x}{\lambda}\right)^{-(\beta+1)\rho} dx\right)^\rho\right]$$

by substituting  $z = (1 + x) - \beta$  and using the series representation,

$$\alpha^{-z} = \sum_{k=1}^{\infty} \frac{(-\log \alpha)^k}{k!} z^k$$

Equation 14

Then we have

$$\begin{aligned} e_R(\rho) &= \frac{1}{1-\rho} \log\left[\left(\frac{\alpha \log \alpha}{\alpha-1}\right)^\rho \frac{\beta^\rho}{\lambda^\rho} \int_0^1 \alpha^{-z\rho} \left(1 + \frac{\lambda(z^{\frac{-1}{\beta}} - 1)}{\lambda}\right)^{-(\beta+1)\rho} \frac{\lambda}{\beta} z^{\frac{-1}{\beta}-1} dz\right] \\ &= \frac{1}{1-\rho} \log\left[\left(\frac{\alpha \log \alpha}{\alpha-1}\right)^\rho \frac{\beta^\rho}{\lambda^\rho} \frac{\lambda}{\beta} \int_0^1 \alpha^{-z\rho} z^{(1+\frac{1}{\beta})\rho} z^{\frac{-1}{\beta}-1} dz\right] \\ &= \frac{1}{1-\rho} \log\left[\left(\frac{\alpha \log \alpha}{\alpha-1}\right)^\rho \frac{\beta^\rho}{\lambda^\rho} \frac{\lambda}{\beta} \sum_{k=0}^{\infty} \frac{(-\log \alpha)^k}{k!} \rho^k \int_0^1 z^{k+\rho+\frac{\rho}{\beta}-\frac{1}{\beta}-1} dz\right] \\ &= \frac{\rho}{1-\rho} \log\left(\frac{\alpha \log \alpha}{\alpha-1}\right) + \frac{1}{1-\rho} \log\left[\frac{\beta^\rho \lambda}{\lambda^\rho \beta} \sum_{k=0}^{\infty} \frac{(-\log \alpha)^k}{k!} \rho^k \frac{\beta}{\beta k + \beta \rho + \rho - 1}\right] \end{aligned}$$

Hence, the theorem is proved.

## 5. THE STRESS-STRENGTH RELIABILITY

In literature related to engineering, reliability, and bio statistics, the stress-strength model has attracted a lot of interest. Let  $X_1$  represents stress and  $X_2$  represents strength. The stress-strength model's most basic version states that a system failure occurs when the stress exceeds the strength. [Kotz and Pensky \(2003\)](#). The stress-strength reliability for the APL is given by the ensuing theorem.

Theorem 5.1. The form of APL's stress-strength reliability

$$R = \frac{\alpha_1 \log \alpha_1}{(\alpha_1 - 1)(\alpha_2 - 1)} \sum_{i=1}^{\infty} \frac{(-\log \alpha_1)^i}{i!} \left[ \alpha_2 \sum_{j=1}^{\infty} \frac{(-\log \alpha_2)^j}{j!(i+j+1)} - \frac{1}{i+1} \right], \quad \alpha_1, \alpha_2 > 0. \tag{Equation 15}$$

Proof. The stress-strength reliability can be defined as

$$S = \int_{-\infty}^{\infty} f_1(x)F_2(x)dx, \tag{Equation 16}$$

using the Equation 5 and Equation 6 of APL distribution, stress strength R, can be obtained as

$$R = \frac{\beta \alpha_1 \log \alpha_1}{\lambda(\alpha_1 - 1)(\alpha_2 - 1)} \int_0^{\infty} \alpha_1^{-(1+\frac{x}{\lambda})^{-\beta}} \left(1 + \frac{x}{\lambda}\right)^{-(\beta+1)} \left(\alpha_2^{1-(1+\frac{x}{\lambda})^{-\beta}} - 1\right) dx$$

by substituting  $z = (1 + x) -\beta$  and using the series representation

$$\alpha_1^{-z} = \sum_{i=1}^{\infty} \frac{(-\log \alpha_1)^i}{i!} z^i, \tag{Equation 17}$$

$$\begin{aligned} R &= \frac{\beta \alpha_1 \log \alpha_1}{\lambda(\alpha_1 - 1)(\alpha_2 - 1)} \int_0^1 \alpha_1^{-z} \left(1 + \frac{\lambda(z^{-\frac{1}{\beta}} - 1)}{\lambda}\right)^{-(\beta+1)} [\alpha_2 \alpha_2^{-z} - 1] \frac{\lambda}{\beta} z^{-\frac{1}{\beta}-1} dz \\ &= \frac{\alpha_1 \log \alpha_1}{(\alpha_1 - 1)(\alpha_2 - 1)} \int_0^1 \alpha_1^{-z} [\alpha_2 \alpha_2^{-z} - 1] dz \\ &= \frac{\alpha_1 \log \alpha_1}{(\alpha_1 - 1)(\alpha_2 - 1)} \sum_{i=1}^{\infty} \frac{(-\log \alpha_1)^i}{i!} \left[ \alpha_2 \sum_{j=1}^{\infty} \frac{(-\log \alpha_2)^j}{j!} \int_0^1 z^{i+j} - z^i dz \right] \\ &= \frac{\alpha_1 \log \alpha_1}{(\alpha_1 - 1)(\alpha_2 - 1)} \sum_{i=1}^{\infty} \frac{(-\log \alpha_1)^i}{i!} \left[ \alpha_2 \sum_{j=1}^{\infty} \frac{(-\log \alpha_2)^j}{j!(i+j+1)} - \frac{1}{i+1} \right] \end{aligned}$$

Hence, the theorem is proved.

### 6. ORDER STATISTICS OF APL

Consider  $x_1 \leq x_2 \leq x_3 \leq \dots \leq x_n$  denotes the order statistic of a random sample of size n from APL distribution with CDF, F(x), and PDF, f(x), the PDF of  $X_r$  is given by Balakrishnan and Cohen (2014)

$$f_r(x) = \frac{n!}{(r-1)!(n-r)!} [F(x)]^{r-1} f(x) [1 - F(x)]^{n-r}, \quad r = 1, 2, n \tag{Equation 18}$$

By using Equation 5, Equation 6 the PDF the rth order statistic from APLD is given by:



$$f(x) = \frac{n! \log a}{(r-1)!(n-r)!(a-1)^n} (a^{1-(1+\frac{x}{\lambda})^{-\beta}} - 1)^{r-1} \frac{\beta}{\lambda} a^{1-(1+\frac{x}{\lambda})^{-\beta}} (1+\frac{x}{\lambda})^{-(\beta+1)} (a^{-(1+\frac{x}{\lambda})^{-\beta}})^{n-r}, x_r > 0.$$

Equation 19

The PDF of the smallest order statistic,  $X_1$ , is as follows

$$f(x) = \frac{n! \log a}{(n-1)!(a-1)^n} \frac{\beta}{\lambda} a^{1-(1+\frac{x}{\lambda})^{-\beta}} (1+\frac{x}{\lambda})^{-(\beta+1)} (a^{-(1+\frac{x}{\lambda})^{-\beta}})^{n-1}, x_1 > 0.$$

Equation 20

And the PDF of the largest order statistic,  $X_n$ , is given by

$$f(x) = \frac{n! \log a}{(a-1)^n} (a^{1-(1+\frac{x}{\lambda})^{-\beta}} - 1)^{n-1} \frac{\beta}{\lambda} a^{1-(1+\frac{x}{\lambda})^{-\beta}} (1+\frac{x}{\lambda})^{-(\beta+1)}, x_n > 0.$$

Equation 21

## 7. METHODS OF ESTIMATION

In the section, we will study classical estimation methods to estimate unknown parameters of the APL distribution. Estimation methods that used are: (1) maximum likelihood method, (2) Ordinary least squares, (3) Weighted least squares.

### 7.1. MAXIMUM LIKELIHOOD ESTIMATION OF APL

Let  $X_1, X_2, \dots, X_n$  be a random sample of size  $n$  from the APL distribution. By using (6), we obtain the likelihood function  $L(x, \alpha, \lambda, \beta)$  as

$$L(x) = (a, \lambda, \beta) = \left(\frac{a \log a}{a-1}\right)^n \prod_{i=1}^n a^{-(1+\frac{x_i}{\lambda})^{-\beta}} \left(\frac{\beta}{\lambda}\right)^n \prod_{i=1}^n (1+\frac{x_i}{\lambda})^{-(\beta+1)},$$

Equation 22

and the log-likelihood function is

$$\log L = n \log\left(\frac{a \log a}{a-1}\right) - \sum_{i=1}^n (1+\frac{x_i}{\lambda})^{-\beta} \log a + n(\log \beta - \log \lambda) - (\beta+1) \sum_{i=1}^n \log\left(1+\frac{x_i}{\lambda}\right).$$

Equation 23

$$= n(\log a \log a - \log a - 1) - \sum_{i=1}^n (1+\frac{x_i}{\lambda})^{-\beta} \log a + n(\log \beta - \log \lambda) - (\beta+1) \sum_{i=1}^n \log\left(1+\frac{x_i}{\lambda}\right).$$

Equation 24

Then by deriving the log-likelihood function with respect to  $\alpha, \lambda$  and  $\beta$  we get

$$\frac{\partial \log L}{\partial a} = n\left(\frac{1 + \log a}{a \log a} - \frac{1}{a-1}\right) - \frac{\sum_{i=1}^n (1+\frac{x_i}{\lambda})^{-\beta}}{a},$$

Equation 25

$$\frac{\partial \log L}{\partial \beta} = \frac{n}{\beta} - \log\left(1 + \frac{x_i}{\lambda}\right) + \left(1 + \frac{x_i}{\lambda}\right)^{-\beta} \log a \log\left(1 + \frac{x_i}{\lambda}\right),$$

Equation 26

$$\frac{\partial \log L}{\partial \lambda} = -\beta \log a \sum_{i=1}^n \left(1 + \frac{x_i}{\lambda}\right)^{-\beta} \frac{x_i}{\lambda^2} - \frac{n}{\lambda} + (\beta + 1) \sum_{i=1}^n \left(\frac{1}{1 + \frac{x_i}{\lambda}}\right) \frac{x_i}{\lambda^2}.$$

Equation 27

After solving Equation 25, Equation 26 and Equation 27 simultaneously using the Newton-Raphson method in Mathematica 12.0, the MLE of  $\alpha$ ,  $\beta$  and  $\lambda$  could be found.

### 7.2. ORDINARY LEAST SQUARES ESTIMATION OF APL

Swain et al. (1988) First suggested the least-squares estimators and weighted least-squares estimators. Assuming a random sample of size  $n$  from the APL distribution, with the observations being in the following order:  $x_{1:n} < x_{2:n} < \dots < x_{n:n}$  we may obtain the ordinary least squares (OLS) estimates  $\alpha$ ,  $\lambda$  and  $\beta$  as follows.

$$S(\alpha, \lambda, \beta) = \sum_{i=1}^n \left[ \frac{\alpha^{1 - (1 + \frac{x_i}{\lambda})^{-\beta}} - 1}{\alpha - 1} - \frac{i}{n + 1} \right]^2$$

Equation 28

Then by deriving the  $S(\alpha, \lambda, \beta)$  function with respect to  $\alpha, \lambda, \beta$ , we get

$$\frac{\partial S(\alpha, \lambda, \beta)}{\partial \alpha} = 2 \sum_{i=1}^n \left[ \frac{\alpha^{1 - (1 + \frac{x_i}{\lambda})^{-\beta}} - 1}{\alpha - 1} - \frac{i}{n + 1} \right] \times \left[ \frac{\alpha^{- (1 + \frac{x_i}{\lambda})^{-\beta}}}{(\alpha - 1)^2} - \frac{\alpha^{1 - (1 + \frac{x_i}{\lambda})^{-\beta}} - 1}{(\alpha - 1)^2} \right]$$

Equation 29

$$\frac{\partial S(\alpha, \lambda, \beta)}{\partial \beta} = 2 \sum_{i=1}^n \left[ \frac{\alpha^{1 - (1 + \frac{x_i}{\lambda})^{-\beta}} - 1}{\alpha - 1} - \frac{i}{n + 1} \right] \times \left[ \frac{\alpha^{1 - (1 + \frac{x_i}{\lambda})^{-\beta}} (1 + \frac{x_i}{\lambda})^{-\beta} \log a \log\left(1 + \frac{x_i}{\lambda}\right)}{(\alpha - 1)} \right]$$

Equation 30

$$\frac{\partial S(\alpha, \lambda, \beta)}{\partial \lambda} = 2 \sum_{i=1}^n \left[ \frac{\alpha^{1 - (1 + \frac{x_i}{\lambda})^{-\beta}} - 1}{\alpha - 1} - \frac{i}{n + 1} \right] \frac{\alpha^{1 - (1 + \frac{x_i}{\lambda})^{-\beta}} x_i \beta (1 + \frac{x_i}{\lambda})^{-\beta} \log a}{(\alpha - 1)^2 \lambda^2}$$

Equation 31

After solving Equation 29, Equation 30, and Equation 31 simultaneously using the Newton-Raphson method in Mathematica 12.0, the OLS of  $\alpha, \beta$  and  $\lambda$  could be obtained.

### 7.3. WEIGHTED LAST SQUARES ESTIMATION OF APL

Weighted least squares (WLS) estimates are represented using the following  $\alpha$ ,  $\beta$  and  $\lambda$  form

$$W(\alpha, \lambda, \beta) = \sum_{i=1}^n \left[ \frac{(n+1)(n+2)}{i(n-i+1)} \right] \left[ \frac{\alpha^{1-(1+\frac{x_i}{\lambda})^{-\beta}} - 1}{\alpha - 1} - \frac{i}{n+1} \right]^2$$

Equation 32

Then by deriving the  $W(\alpha, \lambda, \beta)$  function with respect to  $\alpha, \lambda, \beta$ , we get

$$\frac{\partial W(\alpha, \lambda, \beta)}{\partial \alpha} = 2 \sum_{i=1}^n \left[ \frac{(n+1)(n+2)}{i(n-i+1)} \right] \left[ \frac{\alpha^{1-(1+\frac{x_i}{\lambda})^{-\beta}} - 1}{\alpha - 1} - \frac{i}{n+1} \right] \times \left[ \frac{\alpha^{-(1+\frac{x_i}{\lambda})^{-\beta}} (1 - (1 + \frac{x_i}{\lambda})^{-\beta})}{(\alpha - 1)} - \frac{\alpha^{1-(1+\frac{x_i}{\lambda})^{-\beta}} - 1}{(\alpha - 1)^2} \right]$$

Equation 33

$$\frac{\partial W(\alpha, \lambda, \beta)}{\partial \beta} = 2 \sum_{i=1}^n \left[ \frac{(n+1)(n+2)}{i(n-i+1)} \right] \left[ \frac{\alpha^{1-(1+\frac{x_i}{\lambda})^{-\beta}} - 1}{\alpha - 1} - \frac{i}{n+1} \right] \times \left[ \frac{\alpha^{1-(1+\frac{x_i}{\lambda})^{-\beta}} (1 + \frac{x_i}{\lambda})^{-\beta} \log \alpha \log(1 + \frac{x_i}{\lambda})}{(\alpha - 1)} \right]$$

Equation 34

$$\frac{\partial W(\alpha, \lambda, \beta)}{\partial \lambda} = 2 \sum_{i=1}^n \left[ \frac{(n+1)^2(n+2)}{i(n-i+1)} \right] \left[ \frac{\alpha^{1-(1+\frac{x_i}{\lambda})^{-\beta}} - 1}{\alpha - 1} - \frac{i}{n+1} \right] \times \frac{\alpha^{1-(1+\frac{x_i}{\lambda})^{-\beta}} x_i \beta (1 + \frac{x_i}{\lambda})^{-\beta-1} \log \alpha}{(\alpha - 1)^2 \lambda^2}$$

Equation 35

After solving Equation 33, Equation 34 and Equation 35 simultaneously using the Newton-Raphson method in Mathematica 12.0, the WLS of, and could be obtained.

### 8. SIMULATION STUDY

To make sense of the theoretical findings, simulation using Mathematica 12.0 have been carried out, of the estimating issue presented in the earlier parts. Simulated research has been carried out for mean square error (MSE) and average MLEs.

The estimator computation algorithm is

**Step 1:** Create a random sample of size  $n$  using Equation (9) for the starting parameter values  $(\alpha, \beta, \lambda)$ .

**Step 2:** Use the Newton-Raphson method to solve the equations in to estimate the MLE, LSE and

WLSE of the parameter  $\alpha$  to solve the equations given in (25), (29) and (33), respectively.

**Step 3:** Compute the estimator of entropy from (12), using the estimates in the previous step.

**Step 4:** 1000 times, repeat the first three steps.

**Step 5:** Calculate the MSE.

Different sample sizes of the ML estimate for  $\alpha$ , entropy and MSE for the real parameter values are shown in Table 2. The LSE estimations for  $\alpha$ , entropy and MSE for true parameter values are presented in Table 3. Additionally, the WLSE estimates for  $\alpha$ , entropy, and MSE for the genuine parameter values are shown in Table 4

**Table 2**

Table 2 The MLE and MSE for the Parameters $\alpha$ .					
Parameters	n	Mean ( $\alpha$ )	MSE ( $\alpha$ )	MLE Entropy	MSE Entropy
$\alpha = 2$	30	2.10996	0.055305	0.794999	0.001414
$\beta = 4$	50	2.10507	0.050296	0.794381	0.001292
$\lambda = 3$	200	2.09217	0.049327	0.792177	0.001273
$\rho = 2$	500	2.07664	0.041060	0.789856	0.001079
	1000	2.04933	0.031904	0.785526	0.000860
$\alpha = 3.5$	30	3.49168	0.081989	0.901258	0.000717
$\beta = 3$	50	3.47555	0.080602	0.899818	0.000716
$\lambda = 2$	200	3.47064	0.078377	0.899322	0.000695
$\rho = 2$	500	3.47054	0.074389	0.893239	0.000648
	1000	3.44995	0.065321	0.893229	0.000566
$\alpha = 5.5$	30	5.49820	0.084452	1.815170	0.000363
$\beta = 3$	50	5.49050	0.083072	1.814670	0.000359
$\lambda = 2$	200	5.48660	0.082376	1.814400	0.000352
$\rho = 2$	500	5.47801	0.079926	1.813870	0.000345
	1000	5.47643	0.075352	1.813790	0.000326

**Table 3**

Table 3 The LSE and MSE for the parameters $\alpha$ .					
Parameters	n	Mean ( $\alpha$ )	MSE ( $\alpha$ )	MLE Entropy	MSE Entropy
$\alpha = 2$	30	2.167800	0.059513	0.804842	0.001468
$\beta = 4$	50	2.166970	0.057549	0.804195	0.001427
$\lambda = 3$	200	2.133420	0.045056	0.799589	0.001126
$\rho = 2$	500	2.087780	0.029733	0.792361	0.000760
	1000	2.008250	0.010686	0.779427	0.000294
$\alpha = 3.5$	30	3.464680	0.087933	0.898712	0.000789
$\beta = 3$	50	3.453120	0.083220	0.897577	0.000744
$\lambda = 2$	200	3.385090	0.082102	0.891249	0.000713
$\rho = 2$	500	3.277550	0.079814	0.881001	0.000697
	1000	3.171800	0.051935	0.870616	0.000321
$\alpha = 5.5$	30	5.482420	0.084139	0.885941	0.000360
$\beta = 2$	50	5.457720	0.083646	0.884451	0.000351
$\lambda = 1$	200	5.412630	0.083646	0.881737	0.000321

$\rho = 2$	500	5.307800	0.073992	0.874050	0.000316
	1000	5.192590	0.073594	0.868265	0.000207

**Table 4**

Table 4 The WLSE and MSE for the Parameters $\alpha$					
Parameters	n	Mean ( $\alpha$ )	MSE ( $\alpha$ )	MLE Entropy	MSE Entropy
$\alpha = 2$	30	2.176210	0.062294	0.806236	0.001535
$\beta = 4$	50	2.172040	0.060366	0.805593	0.001495
$\lambda = 3$	200	2.171150	0.059024	0.805502	0.001487
$\rho = 2$	500	2.167290	0.059968	0.804785	0.001462
	1000	2.164850	0.056621	0.804519	0.001402
$\alpha = 3.5$	30	3.476270	0.089437	0.899785	0.000800
$\beta = 3$	50	3.463550	0.087462	0.898549	0.000778
$\lambda = 2$	200	3.46067	0.084138	0.898370	0.000747
$\rho = 2$	500	3.453780	0.083912	0.897605	0.000754
	1000	3.451360	0.082047	0.897472	0.000734
$\alpha = 5.5$	30	5.483850	0.086180	0.886005	0.000312
$\beta = 2$	50	5.483810	0.085978	0.886004	0.000311
$\lambda = 1$	200	5.480090	0.084704	0.885813	0.000308
$\rho = 2$	500	5.474100	0.082359	0.885431	0.000301
	1000	5.458950	0.081502	0.884541	0.000295

From [Table 2](#), [Table 3](#), [Table 4](#), we observed that the MSE of the estimate  $\alpha$  and entropy decreases as the sample size increase. Also, the mean of the estimate  $\alpha$  and the MLE of entropy decreases as the sample size increase.

## 9. APPLICATIONS

In this section. The inverse Lomax distribution (ILD) introduced by [Hassan and Mohamed \(2019\)](#) and the exponential lomax distribution (ELD) introduced by [Abdo et al. \(2015\)](#) will be used to compare the fit of the APLD with other lifetime models that are well-known. Additionally, we take into account the model selection parameters, such as the Akaike information criterion (AIC), log-likelihood ( $\iota$ ), Bayesian information criterion (BIC), consistent Akaike information criterion (CAIC), and Hannan-Quinn information criterion (HQIC). The lowest values of AIC, BIC, CAIC, HQIC and greatest value of,  $\iota$  all point to a better fit of the data, Where

$$AIC = -2\ell(\hat{\theta}) + 2k$$

$$BIC = -2\ell(\hat{\theta}) + k\log n$$

$$CAIC = AIC + \frac{2k(k+1)}{n-k-1}$$

$$HQIC = -2\ell(\hat{\theta}) + 2k\log(\log n)$$

$k$  is the number of parameters;  $n$  is the sample size and  $l(\hat{\theta})$  is the log-likelihood function evaluated at the highest likelihood estimates.

Data I: COVID-19 cases every day in Saudi Arabia

From 1 April to 2 May, this information reveals the daily COVID-19 instances in Saudi Arabia. These statistics were obtained from the Saudi Ministry of Health's website, located at <https://covid19.moh.gov.sa>. These findings are

110, 157, 165, 154, 140, 206, 138, 272, 137, 355, 364, 382, 429, 472, 435, 493, 518, 762, 1132, 1088, 1122,

1147, 1141, 1158, 1172, 1197, 1223, 1289, 1266, 1325, 1351, 1344, 1362.

Table 5 displays descriptive statistics for this data.

**Table 5**

Table 5 Descriptive Statistics for the Number of Daily COVID-19 Cases in Saudi Arabia			
Measure	Value	Measure	Value
N	33	Minimum	110
Maximum	1362	Mean	727.455
Q1	255.5	Q3	1178.25
Median	518.	Skewness	0.0350687
Kurtosis	1.26721	Variance	230684.0
Standard deviation	480.295		

The performance of the APLD in comparison to other models is shown in Table 6 along with the MLEs for the model parameters.

**Table 6**

Table 6 MLEs of the Model Parameters and the Statistics of the AIC, BIC, CAIC, HQIC and for the Number of Daily COVID-19 Cases in Saudi Arabia							
Distributions	MLE $\alpha$	MLE entropy	$l$	AIC	BIC	CAIC	HQIC
APLD	34.7454	2.73066	-243.261	492.532	497.012	493.350	494.033
ELD	0.04329	0.39915	-425.628	857.257	861.746	858.085	858.768
ILD	182.650	2.95628	-253.035	512.070	516.560	512.898	513.581

Table 6 shows that, among the models taken into consideration, the APLD has the smallest AIC, CAIC, BIC, and HQIC values. Which shows that the APLD seems to be a model that might work well with the data set.

Data II: (Natural phenomena) Badr (2019) presented the consist of 30 successive March precipitation (in inches) observations.

77, 52, 90, 174, 162, 205, 81, 131, 120, 32, 195, 95, 120, 81, 47, 281, 143, 187, 337, 118, 220, 135, 300,

475, 309, 248, 151, 96, 210, 189.

Table 7 displays descriptive statistics for this data.

**Table 7**

<b>Table 7 Descriptive Statistics for the of 30 Successive March Precipitation (in inches)</b>			
Measure	Value	Measure	Value
N	30	Minimum	32
Maximum	475	Mean	168.7
Q1	95.	Q3	210.
Median	147.	Skewness	1.12002
Kurtosis	4.30607	Variance	9786.15
Standard deviation	98.925		

The performance of the APLD in comparison to other models is shown in [Table 8](#) along with the MLEs for the model parameters.

**Table 8**

<b>Table 8 MLEs of the Model Parameters and the Statistics of the AIC, BIC, CAIC, HQIC and for the of 30 Successive March Precipitation (In Inches)</b>							
Distributions	MLE $\alpha$	MLE entropy	$\rho$	AIC	BIC	CAIC	HQIC
APLD	66.161	2.83245	-200.811	401.622	412.201	408.422	409.183
ELD	0.0370132	0.351725	-476.589	959.178	963.757	959.978	960.739
ILD	440.78	3.040180	-296.266	598.532	603.111	599.322	600.093

[Table 8](#) is results show that while,  $\rho$  has the largest value and the APLD has the smallest AIC, CAIC, BIC, and HQIC values. As a result, the APLD performs better than other models.

## CONFLICT OF INTERESTS

None.

## ACKNOWLEDGMENTS

None.

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