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ON SOME NON-DERANGED PERMUTATION: A NEW METHOD OF CONSTRUCTION

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Abstract

In this paper, we construct a permutation group via a composition operation on some permutations generated from the structure $|\omega_i + \omega_j| mod p$ for prime $p \geq 5$ and $i \neq j$ as defined by [1]. Thus, providing a new method of constructing permutation group from existing ones.

Keywords: Permutation, Aunu permutation ω_i and Group.

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1. Introduction

Catalan sequence like the other sequences (e.g. Fibonacci and Pell sequences) has been one among the most celebrated sequence and this can be attributed to the astonishing results that have prompt out from the sequence. Martin Gardner describe the sequence as "they have the delightful property for popping up unexpectedly, particularly in combinatorial problems" [2].

The study of Catalan number fully surfaced after the problem of triangulation of regular polygons using the diagonals was resolved. The number (Catalan number) was described in the 18th century by Leonhard Euler but named after the Belgian Mathematician Eugene Charles Catalan [3]. The sequence is given by $C_n = \frac{1}{n+1}\binom{2n}{n}$.

A prominent area in which the application of this sequence is seen is in permutation. According to [4] permutations of length n that avoids any fixed pattern of size three are counted by the Catalan number.

Permutation been a mathematical technique used to determine the number of ways an object can be arranged is governed by specific rules called pattern. Permutations designed to obey peculiar pattern are said to contain the pattern otherwise are said to avoid the pattern. The existence of permutation pattern such as the vincular pattern, the bivincular pattern, the mesh pattern, the

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Alternating pattern, and barred pattern to mention but few, subjected the study of permutations to an area of combinatorics; the permutation statistics.

Permutation group sometimes referred to as symmetric group, is the group of permutations whose permutations under a composition operation form a group. The symmetric group S_n has n! as the number of ways in which the set {1,2,….,n} can be arranged i.e. is the group of all permutations of the set {1,2,….,n}.

Aunu pattern ω_i named after the founders Audu and Aminu [5] is a permutation developed using the Catalan sequence. [6] uses the Catalan numbers on a scheme of prime p: $5 \leq p \leq 11$ and $\Omega \subseteq \mathbb{N}$ to generate cycles of the Aunu permutations ω_i. This was possible via the modular operation mod p. [7] Show that the pattern $\omega_i \in G_p$ for prime $p \geq 5$ is a group by embedding the identity $\{1\}$ in ω_i. The classical pattern 123 and 132 of the Aunu permutations were also studied see [8], [9], [10], [11], [12] and [13].

[14] represented the Aunu pattern as $G_p^{\Gamma_1}$ and the characteristics results of the representation were established. [15] established that every young tableaux of the Aunu permutation group is not standard. [16] define an indiscrete topology on $\mathbb{G}_p'$ and show that the group $\mathbb{G}_p'$ is a toplogical group. [1] define a discrete topology τ on ω_i and study some topological properties of (ω_i, τ). [17] show that the permutation ω_i is Eulerian in its distribution.

Herein, we constructed a group of non-deranged permutations using the enhanced permutations $\left|\omega_i + \omega_j\right| modp \setminus\{2\}$.

2. Terminology

Definition of basic terms are given below:
Definition 1: A *permutation* α on a set $X = \{1,2, \dots, n\}$ is the bijection $\pi(i): X \to X$. it is represented in two line notion as: $\pi(i) = \begin{pmatrix} 1 & 2 & \cdots\cdots & n \\ \pi(1) & \pi(2) & \cdots\cdots & \pi(n) \end{pmatrix}$ and in one line notation as $\pi(i) = (\pi(1), \pi(2), \dots, \pi(\text{n}))$ for $i = 1,2, \dots, n$.

Example. Let $X = (123)$. Then, $\varphi = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ is a permutation on X.

Permutation usually occur in two forms. Permutations with one or more points fixed and permutations with no point fixed. A permutation that fixes no point is called deranged permutation while non-derangement permutations are permutation with a fixed point (i.e. they fixes one point).

Example: $\varphi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$ (Deranged permutation).
$\varphi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$ (Non-deranged permutation).

Definition 2 (Congruency). If a, b, n are integers, $n > 0$, we said a is congruent to b modulo n if and only if $n|(a - b)$. This is written as $a \equiv b(modn)$.

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Definition 3 (Aunu pattern)**.** *Aunu pattern* is a non-deranged permutation whose first entry one (1) is fixed.

The two line notation of this permutation pattern is commonly known as the Γ_1-non deranged permutation and $G_p^{\Gamma_1}$ as its group. The Aunu permutation pattern was defined on the set $\Omega = \{1,2,\ldots.,p\}$ of primes $p \geq 5$. The permutation has a generating function:

$$\omega_i = (1\ (1+i)^{modp}\ (1+2i)^{modp}\ .\ .\ .\ .\ .(1+(p-1)i)^{modp})$$

for $i = 1,\ 2,..p-1$.

Definition 4 (Binary Operation). Let X be a set. Consider the set S of maps $X \rightarrow X$, for $f, g \in S$ define the composition $g \circ f$ for all $x \in X$ by $(g \circ f)x = g\big(f(x)\big)$, then $\circ$ is a binary operation on S.

Definition 5 (Group). Let G be a nonempty set and $*$ be a binary operation on G such that

$$*: G \times G \rightarrow G.$$

Then $(G,*)$ is a group under the operation $*$ if it satisfies the following axioms:
For all $\mathrm{a}, \mathrm{b} \in \mathrm{G},\ (\mathrm{a} * \mathrm{b})$ is closed.

G is associative. That is for all $\mathrm{a}, \mathrm{b}, \mathrm{c}\ \in \mathrm{G},\ \mathrm{a} * (\mathrm{b} * \mathrm{c}) = (\mathrm{a} * \mathrm{b}) * \mathrm{c}$.
G has an identity id. That is for all $\mathrm{a} \in\ \mathrm{G},\ \mathrm{id}\ * \mathrm{a} = \mathrm{a}\ * \mathrm{id} = \mathrm{a}$.
G has an inverse. That is for any $\mathrm{a} \in \mathrm{G}$, there exist $a^{-1} \in \mathrm{G}$ such that $\mathrm{a} *\ a^{-1} =\ a^{-1} * \mathrm{a} = \mathrm{id}$.
The group is said to be Abelian if its elements commutes i.e. for any $a, \mathrm{b} \in \mathrm{G}$ then $\mathrm{a} * \mathrm{b} = \mathrm{b} * \mathrm{a}$.

Definition 6 (Cyclic Group). A group G is said to be cyclic if $\mathrm{G} =< \mathrm{a} >$, for some $\mathrm{a} \in \mathrm{G}$. The element a is called the generator of G.

3. Method of Construction

The structure $|\omega_i + \omega_i| modp$ was adopted from [1] on which we define;

$$\alpha_k = (2\ \ (2+k)^{modp}\ \ (2+2k)^{modp} \ldots \ldots \ldots \ldots .. (2+(p-1)k)^{modp})$$

for k={1,2,…..p-1}, as the generating function for the permutations obtained from $|\omega_i + \omega_j| modp$. Which we called the enhanced permutations.

Afterward a composition operation $\alpha_k \circ \beta$ was defined on the permutation sets using the involution β a member of the permutations α_k.

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4. Results

Theorem: For any prime $p \geq 5$, and $\omega_i, \omega_j \in G_p$, the expected numbers of fixed point permutations generated by $|\omega_i + \omega_j| modp$ is $p - 2$.

Proof: By definition, $modn$ partition the integer n into n-classes. From which we can equally say that $|\omega_i + \omega_j| modp$ partition the addition modulus into p-classes. Thus, the permutation set generated by $|\omega_i + \omega_j| modp$ has the cardinality $p - 1$, since $|\omega_i + \omega_j| modp$ for $j = p - i$ is not a permutation as proven by [1].

If we recall, of $n!$ permutations, $(n - 1)!$ fix a particular point. Similarly, we can say, of $(p - 1)$ permutations generated by $|\omega_i + \omega_j| modp$, $p - 1 - 1 = p - 2$ fix a particular point.

Therefore, $|\omega_i + \omega_j| modp$ generate $p - 2$ set of non-deranged permutations.

Theorem: Let $\alpha_k = \{|\omega_i + \omega_j| modp \setminus \{2\}\}$ for k=1,2….,p-1 be permutations generated from $|\omega_i + \omega_j| modp$. Then for any involution $\beta \in \alpha_k$, the composition $\alpha_k \circ \beta = \{s_m\}$ for $m = (k \times n)$ modp with fixed $n \in k$ is a permutation group.

Proof: Let $\alpha_k = \begin{pmatrix} 1 & 2 & \ldots\ldots\ldots\ldots\ldots & p \\ 2 & (2+k)^{modp} & \ldots\ldots\ldots\ldots\ldots & (2+(p-1)k)^{modp} \end{pmatrix}$ define the two line notation of α_k for k={1,2,…..p-1}.

Then there exist an involution $\beta \in \alpha_k$ such that for any $k \in [1, \ldots, p - 1]$, the composition $\alpha_k \circ \beta$ is unique.

Now, Let $S = \{\alpha_k \circ \beta\}$, then for all $s_i, s_j, s_l \in S$.
$s_i \circ s_j$ is closed.

$(s_i \circ s_j) \circ s_l = (s_j \circ s_l) \circ s_i$. (Associativity)
There exist an $s_q \in S$ such that $s_q \circ s_i = s_i \circ s_q$. ($s_q$ is the identity element)
There exist an inverse $s_m \in S$ such that for some s_l, $s_l \circ s_m = s_m \circ s_l = s_q$.
Thus, $(S, \circ)$ is a group.

Example (Illustration)**.** Let take p=5 for $\omega_i, \omega_j \in G_p$, then
$\omega_1 = (12345)$, $\omega_2 = (13524)$, $\omega_3 = (14253)$, $\omega_4 = (15432)$
So that,
$\alpha_k = \{|\omega_i + \omega_j| modp \setminus \{2\}\} = \{(23451), (24135), (25314), (21543)\}$
We see that there is an involution $\beta = (21543)$ so that the composition $\alpha_k \circ \beta$ is

$$s_1 = \alpha_4 \circ \beta = (12345)$$
$$s_2 = \alpha_3 \circ \beta = (52413)$$
$$s_3 = \alpha_2 \circ \beta = (42531)$$
$$s_4 = \alpha_1 \circ \beta = (32154)$$

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So that $S = \{s_1, s_2, s_3, s_4\}$ and $(S,\circ)$ is a group. See the table below.

$\circ$	s_1	s_2	s_3	s_4
s_1	s_1	s_2	s_3	s_4
s_2	s_2	s_1	s_4	s_3
s_3	s_3	s_4	s_1	s_2
s_4	s_4	s_3	s_2	s_1

Proposition: The group is $(S,\circ)$ an Abelian group.
Proof: Let $a, b \in S$. So that, $a \circ b = \{q_1, q_2 \dots, q_p\} \in S$. We see that, $a \circ b = b \circ a \in S$. This implies by definition that $(S,\circ)$ is Abelian.

Proposition: The group is cyclic.
Proof: Let $S = \{s_1, s_2 \dots, s_{p-1}\}$. Observe that, there exist a $s_i \ni S =< s_i >$Thus, $(S,\circ)$ is cyclic.
Remark: The structure $(S,\circ)$ inherit the cyclic property of its parent permutation group G_p.

Conjecture: For any non-deranged permutation group, the permutations are composition of members of a permutation set of the same length to an involution in the permutation set and the involution to itself.

5. Conclusion

In this paper, we construct a non-deranged permutation group that is Abelian and of course subgroup of the symmetric group S_n using the definition$|\omega_i + \omega_j| modp$. This generation method can be applied to permutations of other types.

We recommend for further structural and functional study of $|\omega_i + \omega_j| modp$ and the newly constructed non-deranged permutation.

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