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LATTICE POINTS ON THE HOMOGENEOUS CUBIC EQUATION WITH FOUR UNKNOWNS $x^2 - xy + y^2 + 4w^2 = 8z^3$

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Notation

$t_{m,n} =$ = Polygonal number of rank n with m sides.
$P^m{}_n =$= Three-dimensional figurate number of rank n with m sides.
S_n= Star number.
$RD(n)$ = Rhombic Dodecahedral number of rank n.

ABSTRACT

The Homogeneous cubic equation with four unknowns represented by the equation $x^2 - xy + y^2 + 4w^2 = 8z^3$ is analyzed for its patterns of non zero distinct integral solutions. Here we exhibit four different patterns. In each pattern we can find some interesting relations between the solutions and special numbers like Polygonal number, Three-Dimensional Figurate number, Star number, Rhombic Dodecahedral number etc.

1. INTRODUCTION

Number theory, called the Queen of Mathematics, is a broad and diverse part of Mathematics that developed from the study of the integers. Diophantine equation is one of the oldest branches of Mathematics. Diophantine problems dominated most of the unsolved mathematical problems.

The cubic equation offers an unlimited field of research because of their variety. This paper concerns with an interesting equation , $\mathrm{x}^2 - \mathrm{xy} + \mathrm{y}^2 + 4\mathrm{w}^2 = 8\mathrm{z}^2$ representing a homogeneous cubic equation with four unknowns for finding its infinitely many solutions and some interesting relations between the solutions and special numbers like Polygonal number, Rhombic Dodecahedral number, Star number, Three Dimensional Figurate number.

2. METHOD OF ANALYSIS

Consider the Homogeneous cubic **e**quation, $\mathrm{x}^2 - \mathrm{xy} + \mathrm{y}^2 + 4\mathrm{w}^2 = 8\mathrm{z}^2$ (1)

2.1. PATTERN 1

Introduction of the transformation $x = u + v, y = u - v, w = v$ (2) in (1) leads to

$u^2 + 7v^2 = 8z^3$ (3)

Assume $a^2 + 7b^2 = z$ (4) and write $8 = (1 + i\sqrt{7})(1 - i\sqrt{7})$ (5)

Using (4) and (5) in (3) and using method of factorization, define

$(u + i\sqrt{7}v) = (1 + i\sqrt{7})(a + i\sqrt{7}\, b)^3$ (6)

Equating real and imaginary parts on both sides of (6), we get

$$u = a^3 - 21ab^2 - 21a^2b + 49b^3$$
$$v = a^3 - 21ab^2 + 3a^2b - 7b^3$$

Substituting u and v in (2), we obtain the solutions of (1) as

$$x(a,b) = 2a^3 - 42ab^2 - 18a^2b + 42b^3$$
$$y(a,b) = -24a^2b + 54b^3$$
$$z(a,b) = a^2 + 7b^2$$
$$w(a,b) = a^3 - 21ab^2 - 21a^2b + 49b^3$$

Some properties for the above solution are listed below:

$$x(a,1) - 2w(a,1) + y(a,1) \equiv 23(mod\ 8).$$
$x(a(a+1),1) - 12{P^3}_{a(a+1)} + 12t_{6,a(a+1)} + 116t_{3,a}$ is a multiple of 6.
$$x(1,b) - 2w(1,b) + y(1,b) + 12{P^3}_b + 2t_{8,b} - 12t_{4,b} = 0$$
$8z(1,b) - 2t_{4,b} - 8$ is a Nasty Number
$$2z(a+1,1) - w(a+1,1) + {P^3}_{a+1} - 50t_{3,a+1} - t_{4,a} = -61$$

2.2. PATTERN 2

Introduction of the transformation $x = u + v, y = u - v, w = v$ (7) in (1) leads to

$u^2 + 7v^2 = 8z^3$ (8)

Assume $a^2 + 7b^2 = z$ (9) and write $8 = \frac{(5+i\sqrt{7})(5-i\sqrt{7})}{4}$ (10)

Using (9) and (10) in (8) and using method of factorization, define

$$(u + i\sqrt{7}v) = \frac{(5+i\sqrt{7})}{2}(a + i\sqrt{7}\, b)^3 \tag{11}$$

Equating real and imaginary parts on both sides of (11), we get

$$u = \frac{1}{2}(5a^3 - 105ab^2 - 21a^2b + 49b^3) \tag{12}$$

$$v = \frac{1}{2}(a^3 - 21ab^2 + 15a^2b - 35b^3) \tag{13}$$

Since our aim is to find integer values for the solution, put $a = 2A$ and $b = 2B$ in (12) and (13), we get

$$u = (20A^3 - 420AB^2 - 84A^2B + 196B^3) \tag{14}$$

$$v = (4A^3 - 84AB^2 + 60A^2B - 140B^3) \tag{15}$$

Substituting (14) and (15) in (7), we obtain the solutions of (1) as

$$\begin{aligned} x(A,B) &= 6A^3 - 126AB^2 - 36A^2B + 14B^3 \\ y(A,B) &= 4A^3 - 84AB^2 - 6A^2B + 84B^3 \\ z(A,B) &= 4A^2 + 48B^2 \\ w(A,B) &= A^3 - 21AB^2 - 15A^2B - 35B^3 \end{aligned}$$

Some properties for the above solution are listed below:

$$x(A,1) - 2w(A,1) - y(A,1) = 0.$$

$$y(A,1) - 4w(A,1) + 348t_{4,A} \equiv 199(mod 4).$$

$$z(1,B) - w(1,B) - 12P^{72}{}_{B} + 4t_{41,B} - 184t_{4,A} = 0.$$

$6z(A,1) - 144$ is a Nasty Number.

$$x(A,1) - 6RD(A) + 83S_A - 7t_{4,A} \equiv 0(mod 3).$$

2.3. PATTERN 3

Introduction of the transformation $x = u - v, y = u + v, w = v$ (16) in (1) leads to

$$u^2 + 7v^2 = 8z^3 \tag{17}$$

Assume $a^2 + 7b^2 = z$ (18) and write $8 = \frac{(31+i\sqrt{7})(31-i\sqrt{7})}{121}$ (19)

Using(18)and (19)in (17) and using method of factorization, define

$$(u + i\sqrt{7}v) = \frac{(31+i\sqrt{7})}{11}(a + i\sqrt{7}\, b)^3 \tag{20}$$

Equating real and imaginary parts on both sides of (20), we get

$$u = \frac{1}{11}(31a^3 - 651ab^2 - 21a^2b + 343b^3) \tag{21}$$

$$v = \frac{1}{11}(a^3 - 21ab^2 + 93a^2b - 1519b^3) \tag{22}$$

Since our aim is to find integer values for the solutions, put $a = 11A$ and $b = 11B$ in (21) and (22), we get

$$u = (3751A^3 - 78771AB^2 - 2541A^2B + 41503B^3) \quad (23)$$

$$v = (121A^3 - 2541AB^2 + 11253A^2B - 183799B^3) \quad (24)$$

Substituting (23) and (24) in (16), we obtain the solutions of (1) as

$$x(A,B) = 3630A^3 - 76230AB^2 - 13794A^2B + 225302B^3$$
$$y(A,B) = 3872A^3 - 81312AB^2 + 8712A^2B - 142296B^3$$
$$z(A,B) = 121A^2 + 847B^2$$
$$w(A,B) = 121A^3 - 2541AB^2 + 11253A^2B - 183799B^3$$

Some properties for the above solution are listed below:

$$x(A,1) - 30w(A,1) + 2904z(A,1) \equiv 0(mod6).$$
$$z(A,1) - 121t_{4,A} \equiv 0(mod7).$$
$$z(A,A+1) - 847t_{8,A} + 15374t_{4,A} \equiv 0(mod5)$$
$$y(1,A) + 35574\, RD(A) - 4477S_A + 1452t_{4,A} \equiv 0(mod5)$$
$$y(A+1,1) - 32w(A+1,1) + 2904z(A+1,1) \equiv 0(mod10).$$

2.4. PATTERN 4

Introduction of the transformation $x = u - v, y = u + v, w = u$ (25) in (1) leads to

$$5u^2 + 3v^2 = 8z^3\ ; \quad (26)$$

Put $u = \alpha + 3\beta, v = \alpha - 5\beta$ (27) in (26), we get $\alpha^2 + 15\beta^2 = z^3$ (28)

Assume $a^2 + 15b^2 = z$ (29) and write $1 = \frac{(1+i\sqrt{15})(1-i\sqrt{15})}{16}$ (30)

Using (29) and (30) in (28) and using method of factorization, define

$$(\alpha + i\sqrt{15}\beta) = \frac{(1+i\sqrt{15})}{4}(a + i\sqrt{15}\,b)^3 \quad (31)$$

Equating real and imaginary parts on both sides of (31), we get

$$\alpha = \frac{1}{4}(a^3 - 45ab^2 - 45a^2b + 225b^3) \quad (32)$$

$$\beta = \frac{1}{4}(a^3 - 45ab^2 + 3a^2b - 15b^3) \quad (33)$$

Since our aim is to find integer values for the solutions, put $a = 4A$ and $b = 4B$ in (32) and (33), we get

$$\alpha = (2A^3 - 90AB^2 - 90A^2B + 4503B^3) \quad (34)$$

$$\beta = (2A^3 - 90AB^2 + 6A^2B - 30B^3) \quad (35)$$

Substituting (34) and (35) in (25)& (26), we obtain the solutions of (1) as

$$x(A,B) = 16A^3 - 720AB^2 + 48A^2B - 240B^3$$
$$y(A,B) = -192A^2B + 960B^3$$
$$z(A,B) = 4A^2 + 60B^2$$

$$w(A,B) = -8A^3 + 360AB^2 - 120A^2B + 600B^3$$

Some properties for the above solution are listed below:

$$y(A,1) + 48z(A,1) \equiv 0(mod2).$$
$$x(A,1) - 2w(A,1) + y(A,1) = 0.$$
$$z(1,B) - 4 \text{, a Nasty Number}$$
$$y(1,B) + 4x(1,B) + 48z(1,B) \text{, a Perfect Square}$$
$$48z(A,1) - x(A,1) + 2w(A,1) \equiv 0(mod6)$$

3. CONCLUSION

Diophantine Equation are rich in variety. To conclude, one may search for several other patterns of solutions and their properties.

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CONFLICT OF INTEREST

The author have declared that no competing interests exist.

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