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SOME INCLUSION PROPERTIES FOR CERTAIN K-UNIFORMLY SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH WRIGHT FUNCTION

E. E. Ali *1
*1 Department of Mathematics and Computer Science, Faculty of Science, Port Said University, Port Said 42521, Egypt

Abstract

A new operator $\mathbf{W}_{\lambda,\mu}^{\alpha} f(z) = z + \sum_{n\geq 2} \frac{4^{n-1}(\alpha)_{(n-1)}\Gamma(\lambda(n-1)+\mu)}{\Gamma(\mu)} a_n z^n$ is introduced for functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in the open unit disk $\mathbf{U} = \{z \in \mathrm{C} : |z| < 1\}$. We introduce several inclusion properties of the new k-uniformly classes $US^*(\alpha;k;\gamma)$, $UC(\alpha;k;\gamma)$, $UK(\alpha;k;\gamma,\beta)$ and $UK^*(\alpha;k;\gamma,\beta)$ of analytic functions defined by using the Wright function with the operator $\mathbf{W}_{\lambda,\mu}^{\alpha}$ and the main object of this paper is to investigate various inclusion relationships for these classes. In addition, we proved that a special property is preserved by some integral operators.

Keywords: Analytic Functions; K-Uniformly Starlike Functions; K-Uniformly Convex Functions; K-Uniformly Close-To-Convex Functions; K-Uniformly Quasi-Convex Functions; Hadamard Product; Subordination.

2000 Mathematics Subject Classification: 30C45; 30D30; 33D20.

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1. Introduction

Let $\mathbf{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

Which are analytic in the open unit disk $\mathbf{U} = \{z \in \mathrm{C} : |z| < 1\}$. If f and g are analytic in $\mathbf{U}$, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz

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function ω, analytic in $\mathbf{U}$ with $\omega(0)=0$ and $|\omega(z)|<1$ $(z\in\mathbf{U})$, such that $f(z)=g(\omega(z))$ $(z\in\mathbf{U})$. In particular, if the function g is univalent in $\mathbf{U}$, the above subordination is equivalent to $f(0)=g(0)$ and $f(\mathbf{U})\subset g(\mathbf{U})$ $(\text{see}\,[9]\text{ and }[10])$.

For functions $f(z)\in\mathbf{A}$, given by (1.1), and $g(z)\in\mathbf{A}$ defined by

$$g(z)=z+\sum_{k=2}^{\infty}b_k z^k,$$

then the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f*g)(z)=z+\sum_{k=2}^{\infty}a_k b_k z^k=(g*f)(z) \qquad (z\in U).$$

For $0\le\gamma,\beta<1$, we denote by $S^*(\gamma)$, $C(\gamma)$, $K(\gamma,\beta)$ and $K^*(\gamma,\beta)$ the subclasses of $\mathbf{A}$ consisting of all analytic functions which are, respectively, starlike of order γ, convex of order γ, close-to-convex of order γ, and type β and quasi-convex of order γ, and type β in $\mathbf{U}$.

Now, we introduce the subclasses $US^*(k;\gamma)$, $UC(k;\gamma)$, $UK(k;\gamma,\beta)$ and $UK^*(k;\gamma,\beta)$ of the class $\mathbf{A}$ for $0\le\gamma,\beta<1$, and $k\ge 0$, which are defined by

$$US^*(k;\gamma)=\left\{f\in\mathbf{A}\ :\Re\left(\frac{zf'(z)}{f(z)}-\gamma\right)>k\left|\frac{zf'(z)}{f(z)}-1\right|\right\}, \tag{1.2}$$

$$UC(k;\gamma)=\left\{f\in\mathbf{A}\ :\Re\left(1+\frac{zf''(z)}{f'(z)}-\gamma\right)>k\left|\frac{zf''(z)}{f'(z)}\right|+\gamma\right\}, \tag{1.3}$$

$$UK(k;\gamma,\beta)=\left\{f\in\mathbf{A}\ :\exists\, g\in US^*(k;\beta)\ s.t.\ \Re\left(\frac{zf'(z)}{g(z)}-\gamma\right)>k\left|\frac{zf'(z)}{g(z)}-1\right|\right\}, \tag{1.4}$$

$$UK^*(k;\gamma,\beta)=\left\{f\in\mathbf{A}\ :\exists\, g\in UC(k;\gamma)\ s.t.\Re\left(\frac{(zf'(z))'}{g'(z)}-\gamma\right)>k\left|\frac{(zf'(z))'}{g'(z)}-1\right|\right\}. \tag{1.5}$$

We note that

$$\left.\begin{array}{c} US^*(0;\gamma)=S^*(k;\gamma),\ UC(0;\gamma)=C(\gamma), \\ UK(0;\gamma,\beta)=K(\gamma,\beta),\ UK^*(0;\gamma,\beta)=K^*(\gamma,\beta) \quad (0\le\gamma,\beta<1). \end{array}\right)$$

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Corresponding to a conic domain $\Omega_{k,\gamma}$ defined by

$$\Omega_{k,\gamma} = \left\{ u+iv \ : \ u > k\sqrt{(u-1)^2+v^2}+\gamma \right\} \tag{1.6}$$

we define the function $q_{k,\gamma}(z)$ which maps $\mathbf{U}$ onto the conic domain $\Omega_{k,\gamma}$ such that $1 \in \Omega_{k,\gamma}$ as the following:

$$q_{k,\gamma}(z) = \begin{cases} \frac{1+(1-2\gamma)z}{1-z} & (k=0), \\ \frac{1-\gamma}{1-k^2}\cos\left\{\frac{2}{\pi}\left(\cos^{-1}k\right)i\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right\} - \frac{k^2-\gamma}{1-k^2} & (0<k<1), \\ 1+\frac{2(1-\gamma)}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 & (k=1), \\ \frac{1-\gamma}{k^2-1}\sin\left\{\frac{\pi}{2\zeta(k)}\int_0^{\frac{u(z)}{\sqrt{k}}}\frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}\right\} + \frac{k^2-\gamma}{k^2-1} & (k>1). \end{cases} \tag{1.7}$$

Where $u(z) = \frac{z-\sqrt{k}}{1-\sqrt{kz}}$ and $\zeta(k)$ is such that $k = \cosh\frac{\pi\zeta'(z)}{4\zeta(z)}$. By virtue of the properties of the conic domain $\Omega_{k,\gamma}$, we have

$$\Re\{q_{k,\gamma}(z)\} > \frac{k+\gamma}{k+1}. \tag{1.8}$$

Making use of the principal of subordination and the definition of $q_{k,\gamma}(z)$, we may rewrite the subclasses $US^*(k;\gamma)$, $UC(k;\gamma)$, $UK(k;\gamma,\beta)$ and $UK^*(k;\gamma,\beta)$ as the following:

$$US^*(k;\gamma) = \left\{ f \in \mathbf{A} \ : \ \frac{zf'(z)}{f(z)} \prec q_{k,\gamma}(z) \right\}, \tag{1.9}$$

$$UC(k;\gamma) = \left\{ f \in \mathbf{A} \ : \ 1+\frac{zf''(z)}{f'(z)} \prec q_{k,\gamma}(z) \right\}, \tag{1.10}$$

$$UK(k;\gamma,\beta) = \left\{ f \in \mathbf{A} \ : \exists\, g \in US^*(k;\beta) \ s.t. \ \frac{zf'(z)}{g(z)} \prec q_{k,\gamma}(z) \right\}, \tag{1.11}$$

$$UK^*(k;\gamma,\beta) = \left\{ f \in \mathbf{A} \ : \exists\, g \in UC(k;\gamma) \ s.t. \frac{(zf'(z))'}{g'(z)} \prec q_{k,\gamma}(z) \right\}. \tag{1.12}$$

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We consider the following normalized form

$$\mathtt{W}_{\lambda,\mu}(z)=\Gamma(\mu)z\mathtt{W}_{\lambda,\mu}(\frac{z}{4})=\sum_{n\geq 0}\frac{\Gamma(\mu)z^{n+1}}{4^{n}n!\Gamma(\lambda n+\mu)},$$

where $\lambda \geq -1, \mu \in \mathrm{C}\setminus \mathrm{Z}_0^-,\ z\in U$. Note that the normalized wright function $\mathtt{W}_{\lambda,\mu}$ was studied recently in [13].

Now, we define an operator $\mathtt{W}_{\lambda,\mu}$ as follows:

$$\mathtt{W}_{\lambda,\mu}f(z)=\mathtt{W}_{\lambda,\mu}(z)*f(z)=z+\sum_{n\geq 2}\frac{\Gamma(\mu)}{4^{n-1}(n-1)!\Gamma(\lambda(n-1)+\mu)}a_n z^n, \tag{1.13}$$

where $\lambda \geq -1, \mu \in \mathrm{C}\setminus \mathrm{Z}_0^-,\ z\in U$. Note that, if $f(z)=\frac{z}{(1-z)}$ then the operator $\mathtt{W}_{\lambda,\mu}f(z)$ reduces to the functions

$$\mathtt{W}_{\nu,b,c}(z)=\mathtt{W}_{1,\nu+\frac{b+1}{2}}(-cz)*\frac{z}{1-z}=(-c)2^{\nu}\Gamma(\nu+(b+1)/2)z^{1-\nu/2}\mathtt{W}_{\nu,b,c}(\sqrt{z}),$$

(1.1)4 $$g_{\nu}(z)=\mathtt{W}_{1,\nu+1}(-z)*\frac{z}{1-z}=(-1)2^{\nu}\Gamma(\nu+1)/2)z^{1-\nu/2}\mathtt{J}_{\nu}(\sqrt{z})$$

And

$$\mathtt{K}_{\nu}(z)=\mathtt{W}_{1,\nu+1}(\frac{z}{1-z})=\mathtt{W}_{1,\nu+1}(z)*\frac{z}{1-z}=2^{\nu}\Gamma(\nu+1))z^{1-\nu/2}\mathtt{I}_{\nu}(\sqrt{z}). \tag{1.15}$$

Note that the function $\upsilon_{\nu,b,c}(z)$ was studied recently in [1, 2, 11] and $\mathrm{g}_{\nu}(z)$ was investigated in [3, 12, 14].

Corresponding to the function $\mathtt{W}_{\lambda,\mu}(z)$ defined by (1.13), we introduce a function $\mathtt{W}_{\lambda,\mu}^{\alpha}(z)$ given by

$$\mathtt{W}_{\lambda,\mu}(z)*\mathtt{W}_{\lambda,\mu}^{\alpha}(z)=\frac{z}{(1-z)^{\alpha}}\quad (\alpha>0)\,. \tag{1.16}$$

We now define an operator $\mathtt{W}_{\lambda,\mu}^{\alpha}f(z):\ \mathtt{A}\rightarrow\mathtt{A}$ by

$$\mathtt{W}_{\lambda,\mu}^{\alpha}f(z)=\mathtt{W}_{\lambda,\mu}^{\alpha}(z)*f(z) \tag{1.17}$$

$\left(\lambda \geq -1, \mu \in \mathrm{C}\setminus \mathrm{Z}_0^-,\ \alpha>0\,; z\in U\right)$.

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If $f(z)$ is given by (1.1) , then from (1.17) , we deduce that

$$\mathbf{W}_{\lambda,\mu}^{\alpha} f(z) = z + \sum_{n\geq 2} \frac{4^{n-1}(\alpha)_{(n-1)}\Gamma(\lambda(n-1)+\mu)}{\Gamma(\mu)} a_n z^n \qquad (1.18)$$
$$(\lambda \geq -1, \mu \in \mathbb{C}\setminus \mathbb{Z}_0^-,\ \alpha > 0\,; z \in U).$$

It is easily to deduce from (1.18) that.

$$z\left(\mathbf{W}_{\lambda,\mu}^{\alpha} f(z)\right)' = \alpha\left(\mathbf{W}_{\lambda,\mu}^{\alpha+1} f(z)\right) - (\alpha-1)\left(\mathbf{W}_{\lambda,\mu}^{\alpha} f(z)\right). \qquad (1.19)$$

Next, by using the operator $\mathbf{W}_{\lambda,\mu}^{\alpha}$, we introduce the following classes of analytic functions for $\lambda \geq -1, \mu \in \mathbb{C}\setminus \mathbb{Z}_0^-,\ \alpha > 0,\quad k \geq 0$ and $0 \leq \gamma, \beta < 1$:

$$US^*(\alpha;k;\gamma) = \left\{ f \in \mathbf{A} \, : \, \mathbf{W}_{\lambda,\mu}^{\alpha} f(z) \in US^*(k;\gamma) \right\}, \qquad (1.20)$$

$$UC(\alpha;k;\gamma) = \left\{ f \in \mathbf{A} \, : \, \mathbf{W}_{\lambda,\mu}^{\alpha} f(z) \in UC(k;\gamma) \right\}, \qquad (1.21)$$

$$UK(\alpha;k;\gamma,\beta) = \left\{ f \in \mathbf{A} \, : \, \mathbf{W}_{\lambda,\mu}^{\alpha} f(z) \in UK(k;\gamma,\beta) \right\}, \qquad (1.22)$$

$$UK^*(\alpha;k;\gamma,\beta) = \left\{ f \in \mathbf{A} \, : \, \mathbf{W}_{\lambda,\mu}^{\alpha} f(z) \in UK^*(k;\gamma,\beta) \right\}. \qquad (1.23)$$

We also note that

$$f(z) \in US^*(\alpha;k;\gamma) \Leftrightarrow zf'(z) \in UC(\alpha;k;\gamma),$$

and

$$f(z) \in UK(\alpha;k;\gamma,\beta) \Leftrightarrow zf'(z) \in UK^*(\alpha;k;\gamma,\beta). \qquad (1.24)$$

In this paper, we investigate several inclusion properties of the classes $US^*(\alpha;k;\gamma)$, $UC(\alpha;k;\gamma)$, $UK(\alpha;k;\gamma,\beta)$ and $UK^*(\alpha;k;\gamma,\beta)$ associated with the operator $\mathbf{W}_{\lambda,\mu}^{\alpha}$ Some applications involving integral operators are also considered.

2. Inclusion Properties Involving the Operator $\mathbf{W}_{\lambda,\mu}^{\alpha}$

In order to prove the main results, we shall need the following lemmas.

Lemma 1 [5]. *Let* $h(z)$ *be convex univalent in* $\mathbf{U}$ *with* $h(0)=1$ *and* $\Re\{\eta h(z)+\gamma\}>0$ $(\eta,\gamma \in \mathbb{C})$. *If* $p(z)$ *is analytic in* $\mathbf{U}$ *with* $p(0)=1$ *, then*

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$$p(z)+\frac{zp'(z)}{\eta p(z)+\gamma}\prec h(z) \tag{2.1}$$

Implies

$$p(z)\prec h(z)\,. \tag{2.2}$$

Lemma 2 [8]. *Let* $h(z)$ *be convex univalent in* $\mathbf{U}$ *and let* w *be analytic in* $\mathbf{U}$ *with* $\Re\{w(z)\}\geq 0$. *If* $p(z)$ *is analytic in* $\mathbf{U}$ *and* $p(0)=h(0)$ *, then*

$$p(z)+w(z)zp'(z)\prec h(z) \tag{2.3}$$

Implies

$$p(z)\prec h(z). \tag{2.4}$$

Theorem 1. $US^{*}(\alpha+1;k;\gamma)\subset US^{*}(\alpha;k;\gamma)$.

Proof. Let $f\in US^{*}(\alpha+1;k;\gamma)$ and set

$$p(z)=\frac{z\left(\mathbf{W}_{\lambda,\mu}^{\alpha}f(z)\right)'}{\mathbf{W}_{\lambda,\mu}^{\alpha}f(z)}\quad (z\in\mathbf{U}), \tag{2.5}$$

where $p(z)$ is analytic in $\mathbf{U}$ with $p(0)=1$. From (1.19) and (2.5), we have

$$\frac{\mathbf{W}_{\lambda,\mu}^{\alpha+1}f(z)}{\mathbf{W}_{\lambda,\mu}^{\alpha}f(z)}=\frac{1}{\alpha}\{p(z)+(\alpha-1)\}. \tag{2.6}$$

Differentiating (2.6) with respect to z and multiplying the result equation by z, we obtain

$$\frac{z(\mathbf{W}_{\lambda,\mu}^{\alpha+1}f(z))'}{\mathbf{W}_{\lambda,\mu}^{\alpha+1}f(z)}=p(z)+\frac{zp'(z)}{p(z)+(\alpha-1)}. \tag{2.7}$$

From this and the argument given in Section 1, we may write

$$p(z)+\frac{zp'(z)}{p(z)+(\alpha-1)}\prec q_{k,\gamma}(z)\quad (z\in\mathbf{U}). \tag{2.8}$$

Since ($\alpha-1)>0$ and $\Re\{q_{k,\gamma}(z)\}>\dfrac{k+\gamma}{k+1}$, we see that

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$$\Re\{q_{k,\gamma}(z)+(\alpha-1)\}>0 \quad (z\in\mathbf{U}). \tag{2.9}$$

Applying Lemma 1 to (2.8), it follows that $p(z)\prec q_{k,\gamma}(z)$, that is, $f\in US^*(\alpha;k;\gamma)$.

Theorem 2. $UC(\alpha+1;k;\gamma)\subset UC(\alpha;k;\gamma)$.

Proof. Applying (1.24) and Theorem 1, we observe that

$$f(z)\in UC(\alpha+1;k;\gamma)\Leftrightarrow zf'(z)\in US^*(\alpha+1;k;\gamma)$$
$$\Rightarrow zf'(z)\in US^*(\alpha;k;\gamma)$$
$$\Leftrightarrow f(z)\in UC(\alpha;k;\gamma),$$

which evidently proves Theorem 2.

Theorem 3. $UK(\alpha+1;k;\gamma,\beta)\subset UK(\alpha;k;\gamma,\beta)$.

Proof. Let $f\in UK(\alpha+1;k;\gamma,\beta)$. Then, from the definition of $UK(\alpha+1;k;\gamma,\beta)$, there exists a function $r(z)\in US^*(k;\gamma)$ such that

$$\frac{z\left(\mathbf{W}_{\lambda,\mu}^{\alpha+1}f(z)\right)'}{r(z)}\prec q_{k,\gamma}(z). \tag{2.10}$$

Choose the function $g(z)$ such that $\mathbf{W}_{\lambda,\mu}^{\alpha+1}g(z)=r(z)$. Then, $g\in US^*(\alpha+1;k;\gamma)$ and

$$\frac{z\left(\mathbf{W}_{\lambda,\mu}^{\alpha+1}f(z)\right)'}{\mathbf{W}_{\lambda,\mu}^{\alpha+1}g(z)}\prec q_{k,\gamma}(z). \tag{2.11}$$

Now let

$$p(z)=\frac{z\left(\mathbf{W}_{\lambda,\mu}^{\alpha}f(z)\right)'}{\mathbf{W}_{\lambda,\mu}^{\alpha}g(z)}, \tag{2.12}$$

where $p(z)$ is analytic in $\mathbf{U}$ with $p(0)=1$. Since $g\in US^*(\alpha+1;k;\gamma)$, by Theorem 1, we know that $g\in US^*(\alpha;k;\gamma)$. Let

$$t(z)=\frac{z\left(\mathbf{W}_{\lambda,\mu}^{\alpha}g(z)\right)'}{\mathbf{W}_{\lambda,\mu}^{\alpha}g(z)} \quad (z\in\mathbf{U}), \tag{2.13}$$

where $t(z)$ is analytic in $\mathbf{U}$ with $\Re\{t(z)\}>\frac{k+\gamma}{k+1}$. Also, from (2.13), we note that

$$z\left(\mathbf{W}_{\lambda,\mu}^{\alpha}f(z)\right)'=\mathbf{W}_{\lambda,\mu}^{\alpha}zf'(z)=\left(\mathbf{W}_{\lambda,\mu}^{\alpha}g(z)\right)\,p(z). \tag{2.14}$$

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Differentiating both sides of (2.14) with respect to z and multiplying the result equation by z, we obtain

$$\frac{z\left(\mathbf{W}_{\lambda,\mu}^{\alpha} zf'(z)\right)'}{\mathbf{W}_{\lambda,\mu}^{\alpha} g(z)} = \frac{z\left(\mathbf{W}_{\lambda,\mu}^{\alpha} g(z)\right)'}{\mathbf{W}_{\lambda,\mu}^{\alpha} g(z)} p(z) + zp'(z) = t(z)p(z) + zp'(z). \tag{2.15}$$

Now using the identity (1.19) and (2.15), we obtain

$$\frac{z\left(\mathbf{W}_{\lambda,\mu}^{\alpha+1} f(z)\right)'}{\mathbf{W}_{\lambda,\mu}^{\alpha+1} g(z)} = \frac{\mathbf{W}_{\lambda,\mu}^{\alpha+1} zf'(z)}{\mathbf{W}_{\lambda,\mu}^{\alpha+1} g(z)} = \frac{z\left(\mathbf{W}_{\lambda,\mu}^{\alpha} zf'(z)\right)' + (\alpha-1)\mathbf{W}_{\lambda,\mu}^{\alpha} zf'(z)}{z\left(\mathbf{W}_{\lambda,\mu}^{\alpha} g(z)\right)' + (\alpha-1)\mathbf{W}_{\lambda,\mu}^{\alpha} g(z)}$$

$$= \frac{\dfrac{z\left(\mathbf{W}_{\lambda,\mu}^{\alpha} zf'(z)\right)'}{\mathbf{W}_{\lambda,\mu}^{\alpha} g(z)} + (\alpha-1)\dfrac{z\left(\mathbf{W}_{\lambda,\mu}^{\alpha} f(z)\right)'}{\mathbf{W}_{\lambda,\mu}^{\alpha} g(z)}}{\dfrac{z\left(\mathbf{W}_{\lambda,\mu}^{\alpha} g(z)\right)'}{\mathbf{W}_{\lambda,\mu}^{\alpha} g(z)} + (\alpha-1)}$$

$$= \frac{t(z)p(z) + zp'(z) + (\alpha-1)p(z)}{t(z) + (\alpha-1)}$$

$$= p(z) + \frac{zp'(z)}{t(z) + (\alpha-1)}. \tag{2.16}$$

Since $(\alpha-1) > 0$ and $\Re\{t(z)\} > \frac{k+\gamma}{k+1}$, we see that

$$\Re\{t(z) + (\alpha-1)\} > 0 \quad (z \in \mathbf{U}). \tag{2.17}$$

Hence, applying Lemma 2, we can show that $p(z) \prec q_{k,\gamma}(z)$ so that $f \in UK(\alpha;k;\gamma,\beta)$. This completes the proof of Theorem 3.

Theorem 4. $UK^*(\alpha+1;k;\gamma,\beta) \subset UK^*(\alpha;k;\gamma,\beta)$.

Proof. Just as we derived Theorem 2 as consequence of Theorem 1 by using the equivalence (1.24), we can also prove Theorem 4 by using Theorem 3 and the equivalence (1.25).

3. Inclusion Properties Involving the Integral Operator F_c

In this section, we consider the generalized Libera integral operator F_c (see [4, 6, 7]) defined by

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$$F_c(f)(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)\,dt \qquad (f\in\mathbf{A};c>-1). \tag{3.1}$$

Theorem 5. Let $c>-\frac{k+\gamma}{k+1}$. If $f\in US^*(\alpha;k;\gamma)$, then $F_c(f)\in US^*(\alpha;k;\gamma)$.
Proof. Let $f\in US^*(\alpha;k;\gamma)$ and set

$$p(z)=\frac{z\left(\mathbf{W}_{\lambda,\mu}^{\alpha}F_c(f)(z)\right)'}{\mathbf{W}_{\lambda,\mu}^{\alpha}F_c(f)(z)} \quad (z\in\mathbf{U}), \tag{3.2}$$

where $p(z)$ is analytic in $\mathbf{U}$ with $p(0)=1$. From (3.2), we have

$$z\left(\mathbf{W}_{\lambda,\mu}^{\alpha}F_c(f)(z)\right)'=(c+1)\mathbf{W}_{\lambda,\mu}^{\alpha}f(z)-c\mathbf{W}_{\lambda,\mu}^{\alpha}F_c(f)(z). \tag{3.3}$$

Then, by using (3.2) and (3.3), we obtain

$$(c+1)\frac{\mathbf{W}_{\lambda,\mu}^{\alpha}f(z)}{\mathbf{W}_{\lambda,\mu}^{\alpha}F_c(f)(z)}=p(z)+c. \tag{3.4}$$

Taking the logarithmic differentiation on both sides of (3.4) and multiplying by z, we have

$$p(z)+\frac{zp'(z)}{p(z)+c}=\frac{z\left(\mathbf{W}_{\lambda,\mu}^{\alpha}f(z)\right)'}{\mathbf{W}_{\lambda,\mu}^{\alpha}f(z)}\prec q_{k,\gamma}(z). \tag{3.5}$$

Hence, by virtue of Lemma 1, we conclude that $p(z)\prec q_{k,\gamma}(z)$ in $\mathbf{U}$, which implies that $F_c(f)\in US^*(\alpha;k;\gamma)$.

Theorem 6. Let $c>-\frac{k+\gamma}{k+1}$. If $f\in UC(\alpha;k;\gamma)$, then $F_c(f)\in UC(\alpha;k;\gamma)$.
Proof. By applying Theorem 5, it follows that

$$\begin{aligned} f(z)\in UC(\alpha;k;\gamma) &\Leftrightarrow zf'(z)\in US^*(\alpha;k;\gamma)\\ &\Rightarrow F_c(zf')(z)\in US^*(\alpha;k;\gamma) \quad (by\ Theorem\ 5)\\ &\Leftrightarrow z(F_c(f)(z))'\in US^*(\alpha;k;\gamma)\\ &\Leftrightarrow F_c(f)(z)\in UC(\alpha;k;\gamma), \end{aligned} \tag{3.6}$$

which proves Theorem 6.

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Theorem 7. Let $c > -\frac{k+\gamma}{k+1}$. *If* $f \in UK(\alpha;k;\gamma,\beta)$ *, then* $F_c(f) \in UK(\alpha;k;\gamma,\beta)$.

Proof. Let $f \in UK(\alpha;k;\gamma,\beta)$. Then, in view of the definition of the class $UK(\alpha;k;\gamma,\beta)$, there exists a function $g \in US^*(\alpha;k;\gamma)$ such that

$$\frac{z\left(\mathbf{W}_{\lambda,\mu}^{\alpha} f(z)\right)'}{\mathbf{W}_{\lambda,\mu}^{\alpha} g(z)} \prec q_{k,\gamma}(z). \tag{3.7}$$

Thus, we set

$$p(z) = \frac{z\left(\mathbf{W}_{\lambda,\mu}^{\alpha} F_c(f)(z)\right)'}{\mathbf{W}_{\lambda,\mu}^{\alpha} F_c(g)(z)} \quad (z \in \mathbf{U}), \tag{3.8}$$

where $p(z)$ is analytic in $\mathbf{U}$ with $p(0) = 1$. Since $g \in US^*(\alpha;k;\gamma)$, we see from Theorem 5. that $F_c(g) \in US^*(\alpha;k;\gamma)$. Using (3.3) and let

$$t(z) = \frac{z\left(\mathbf{W}_{\lambda,\mu}^{\alpha} F_c(g)(z)\right)'}{\mathbf{W}_{\lambda,\mu}^{\alpha} F_c(g)(z)}, \tag{3.9}$$

where $t(z)$ is analytic in $\mathbf{U}$ with $\Re\{t(z)\} > \frac{k+\gamma}{k+1}$. Using (3.8), we have

$$\mathbf{W}_{\lambda,\mu}^{\alpha} z F_c'(f)(z) = \left(\mathbf{W}_{\lambda,\mu}^{\alpha} F_c(g)(z)\right) p(z). \tag{3.10}$$

Differentiating both sides of (3.10) with respect to z and multiplying by z , we obtain

$$\frac{z\left(\mathbf{W}_{\lambda,\mu}^{\alpha} z F_c'(f)(z)\right)'}{\mathbf{W}_{\lambda,\mu}^{\alpha} F_c(g)(z)} = \frac{z\left(\mathbf{W}_{\lambda,\mu}^{\alpha} F_c(f)(z)\right)'}{\mathbf{W}_{\lambda,\mu}^{\alpha} F_c(g)(z)} p(z) + z p'(z)$$

$$= t(z)p(z) + zp'(z). \tag{3.11}$$

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Now using the identity (3.3) and (3.11), we obtain

$$\frac{z\left(\mathbf{W}_{\lambda,\mu}^{\alpha} f(z)\right)'}{\mathbf{W}_{\lambda,\mu}^{\alpha} g(z)} = \frac{\mathbf{W}_{\lambda,\mu}^{\alpha} zf'(z)}{\mathbf{W}_{\lambda,\mu}^{\alpha} g(z)} = \frac{z\left(\mathbf{W}_{\lambda,\mu}^{\alpha} zF_c'(f)(z)\right)' + c\mathbf{W}_{\lambda,\mu}^{\alpha} zF_c'(f)(z)}{z\left(\mathbf{W}_{\lambda,\mu}^{\alpha} F_c(g)(z)\right)' + c\mathbf{W}_{\lambda,\mu}^{\alpha} F_c(g)(z)}$$

$$= \frac{\dfrac{z\left(\mathbf{W}_{\lambda,\mu}^{\alpha} zF_c'(f)(z)\right)'}{\mathbf{W}_{\lambda,\mu}^{\alpha} F_c(g)(z)} + c\dfrac{z\left(\mathbf{W}_{\lambda,\mu}^{\alpha} F_c(f)(z)\right)'}{\mathbf{W}_{\lambda,\mu}^{\alpha} F_c(g)(z)}}{\dfrac{z\left(\mathbf{W}_{\lambda,\mu}^{\alpha} F_c(g)(z)\right)'}{\mathbf{W}_{\lambda,\mu}^{\alpha} F_c(g)(z)} + c}$$

$$= \frac{t(z)p(z) + zp'(z) + cp(z)}{t(z)+c}.$$

$$= p(z) + \frac{zp'(z)}{t(z)+c} \tag{3.12}$$

Since $c > -\frac{k+\gamma}{k+1}$ and $\Re\{t(z)\} > \frac{k+\gamma}{k+1}$, we see that

$$\Re\{t(z)+c\} > 0 \quad (z \in \mathbf{U}). \tag{3.13}$$

Applying Lemma 2 to (3.12), it follows that $p(z) \prec q_{k,\gamma}(z)$, that is $F_c(f) \in UK(\alpha;k;\gamma,\beta)$.

Theorem 8. Let $c > -\frac{k+\gamma}{k+1}$. *If* $f \in UK^*(\alpha;k;\gamma,\beta)$ *, then* $F_c(f) \in UK^*(\alpha;k;\gamma,\beta)$.

Proof. Just as we derived Theorem 6 as consequence of Theorem 5 and (1.24), we easily deduce the integral-preserving property asserted by Theorem 8 by using Theorem 7 and (1.25).

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*Corresponding author.
E-mail address: ekram_008eg@ yahoo.com