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CONVEXITY AND CONCAVITY OF MEANS

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Abstract

In this paper, convexity and concavity among Greek means are discussed and the results are interpreted with Vander monde's determinant.

Keywords: Convexity; Greek Means; Vander Monde's Determinant; Functions.

1. Introduction

The well –known means are presented by Pappus of Alexandria in his books in fourth century A. D., which is the main contribution of the ancient Greeks. In Pythagorean School on the basis of proportion, ten Greek means are defined, out of which six means are named and four means are un-named. The popular named means are Arithmetic mean, Geometric mean, Harmonic mean and Contra harmonic mean. The un-named Greek means are $F_7(a, b)$, $F_8(a, b)$, $F_9(a, b)$ and $F_{10}(a, b)$ are given below; [13] here we consider the means needed for to develop this paper.

(1.a) $G(a, b) = \frac{a-m}{m-b} = \frac{a}{m} = \frac{m}{b} = \sqrt{ab}$

(1.b) $H(a, b) = \frac{a-m}{m-b} = \frac{a}{b} = \frac{2ab}{a+b}$

(1.c) $F_7(a, b) = \frac{a-m}{m-b} = \frac{b}{a} = \frac{a^2-ab+b^2}{a}$

(1.d) $F_9(a, b) = \frac{a-b}{m-b} = \frac{a}{b} = \frac{b(2a-b)}{a}$

Results on convexity and concavity of one function with respect to another function were in detail discussed in [1] and also some convexity and concavity results on various important means and their applications to mean inequalities were found in [9, 11, 12, 14].

2. Definitions and Lemmas

In this section, we shall recall some definitions and lemmas which is necessary to develop this paper.

Definition 2.a. A mean is a function $M: R_+^2 \to R_+$,
which has the property $a \wedge b \le M(a,b) \le a \vee b$, $\forall a, b > 0$,
where $a \wedge b = min(a,b)$ and $a \vee b = max(a,b)$.

Definition 2.b. Vander Monde's determinant Let Ψ be a continuous function on an interval $I \subseteq R, a = (a_0, a_1, a_2 \ldots \ldots \ldots a_n)$ and $a_i \in I, a_i \ne a_j$ for $i \ne j$ see [8]. Setting

$$(2.1) \qquad V(a, \Psi) = \begin{vmatrix} 1 & a_0 & a_0^2 & \ldots & a_0^{n-1} & \Psi(a_0) \\ 1 & a_1 & a_1^2 & \ldots & a_1^{n-1} & \Psi(a_1) \\ \ldots & & \ldots & \ldots & \ldots & \ldots \\ 1 & a_n & a_n^2 & \ldots & a_n^{n-1} & \Psi(a_n) \end{vmatrix}$$

Let $\Psi(x) = x^{n+r} ln^k x$ in 2.1, we have

$$(2.2) \qquad V(a, \Psi) = \begin{vmatrix} 1 & a_0 & a_0^2 & \ldots & a_0^{n-1} & a_0^{n+r} ln^k a_0 \\ 1 & a_1 & a_1^2 & \ldots & a_1^{n-1} & a_1^{n+r} ln^k a_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & a_n & a_n^2 & \ldots & a_n^{n-1} & a_n^{n+r} ln^k a_n \end{vmatrix}$$

Note that when $r = 0$ and $k = 0$ is the determinant of Vander Monde's matrix of $(n+1)^{th}$ order. The following lemma (2.1) and (2.2) are particular cases of determinant (2.1) or (2.2)

Lemma 2.1. For $\Psi(x) = x^2$ and $a = (a_0, a_1, a_2)$ is the determinant of Vander Monde's matrix of the 3rd order takes the form:

$$(2.3) \qquad V(a; r = 0, k = 0) = \begin{vmatrix} 1 & a_0 & a_0^2 \\ 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \end{vmatrix}$$

which is equivalently
$$(2.4) \qquad V(a; r = 0, k = 0) = (a_1 - a_0)(a_2 - a_0)(a_2 - a_1)$$

Lemma 2.2. For $\Psi(x) = x^{1/2} = \sqrt{x}$ and $a = (a_0, a_1, a_2)$ is the determinant of Vander Monde's matrix of 3rd order takes the form:

$$(2.5) \qquad V(a; r = 0, k = 0) = \begin{vmatrix} 1 & a_0 & \sqrt{a_0} \\ 1 & a_1 & \sqrt{a_1} \\ 1 & a_2 & \sqrt{a_2} \end{vmatrix}$$

which is equivalently
$$(2.6) \qquad V(a; r = -3/2, k = 0) = (\sqrt{a_1} - \sqrt{a_0})(\sqrt{a_2} - \sqrt{a_0})(\sqrt{a_2} - \sqrt{a_1})$$

Lemma 2.3. [1] Let $h(x)$ and $k(x)$ are two functions, then $k(x)$ is said to be convex with respect to $h(x)$ for $a \le b \le c$ if and only if

$$(2.7) \quad \begin{vmatrix} 1 & h(a) & k(a) \\ 1 & h(b) & k(b) \\ 1 & h(c) & k(c) \end{vmatrix} \geq 0 \quad \simeq \quad \begin{vmatrix} 1 & h(a) & k(a) \\ 0 & h(b)-h(a) & k(b)-k(a) \\ 0 & h(c)-h(a) & k(c)-k(a) \end{vmatrix} \geq 0$$

which is equivalently

$$(2.8) \qquad [h(b)-h(a)][k(c)-k(a)] - [h(c)-h(a)][k(b)-k(a)] \geq 0.$$

Setting $a = x$ and $b = 1$ in eqs (1.a) to (1.d), The popular named means Geometric mean, Harmonic mean. The un-named means $F_7(a,b)$ and $F_9(a,b)$ takes the following form:

(i) $G(x,1) = \sqrt{x}$

ii) $H(x,1) = \dfrac{2x}{x+1}$

iii) $F_7(x,1) = \dfrac{x^2-x+1}{x}$

iv) $F_9(x,1) = \dfrac{(2x-1)}{x}$

3. Main Results

In this section, the necessary and sufficient conditions for Convexity and Concavity of different means are discussed and the results are expressed in terms of Vander Monde's determinants.

Theorem 3.1. The Harmonic mean is concave (convex) with respect to $F_7(a,b)$ if and only if $V(a; r = 0, k = 0) \leq (\geq)0$.

Proof: Consider the Harmonic mean and $F_7(a,b)$ in the form;

$$H(x,1) = \frac{2x}{x+1} \text{ and } F_7(x,1) = \frac{x^2-x+1}{x}$$

Let $h(x) = \dfrac{2x}{x+1}$ and $k(x) = \dfrac{x^2-x+1}{x}$, by lemma (2.3) we have

$$\begin{vmatrix} 1 & h(a) & k(a) \\ 0 & h(b)-h(a) & k(b)-k(a) \\ 0 & h(c)-h(a) & k(c)-k(a) \end{vmatrix} = \begin{vmatrix} 1 & \frac{2a}{a+1} & \frac{a^2-a+1}{a} \\ 0 & \frac{2(b-a)}{(a+1)(b+1)} & \frac{b^2-b+1}{b} - \frac{a^2-a+1}{a} \\ 0 & \frac{2(c-a)}{(c+1)(a+1)} & \frac{c^2-c+1}{c} - \frac{a^2-a+1}{a} \end{vmatrix}$$

On Simplifying the determinant leads to
$$\frac{2\,(b-a)(c-a)(c-b)(abc+1)}{2abc(a+1)(b+1)(c+1)} \geq 0.$$

By lemma (2.1) we have

$$(3.1) \qquad \frac{2\,(b-a)(c-a)(c-b)(abc+1)}{2abc(a+1)(b+1)(c+1)} = \frac{2V(a;r=0,k=0)(abc+1)}{2abc(a+1)(b+1)(c+1)} \geq 0$$

By lemma (2.3), $0 < a < b < c$ implies that $2abc(a+1)(b+1)(c+1) \geq 0$

Similarly by considering $h(x) = \dfrac{x^2-x+1}{x}$ and $k(x) = \dfrac{2x}{x+1}$, by lemma (2.3) we have

$$(3.2) \qquad \frac{2\,(b-a)(c-a)(b-c)(abc+1)}{2abc(a+1)(b+1)(c+1)} = \frac{2V(a;r=0,k=0)(abc+1)}{2abc(a+1)(b+1)(c+1)} \leq 0$$

By combining eqs (3.1) and (3.2) the proof of theorem (3.1) completes.

Theorem 3.2. The Harmonic mean is concave (convex) with respect to $F_9(a,b)$ if and only if $V(a; r = 0, k = 0) \leq (\geq)0$.

Proof: Consider the Harmonic mean and $F_9(a,b)$ in the form;

$$H(x,1) = \frac{2x}{x+1} \text{ and } F_9(x,1) = \frac{2x-1}{x}$$

Let $h(x) = \frac{2x}{x+1}$ and $k(x) = \frac{2x-1}{x}$, by lemma (2.3) we have

$$\begin{vmatrix} 1 & h(a) & k(a) \\ 0 & h(b)-h(a) & k(b)-k(a) \\ 0 & h(c)-h(a) & k(c)-k(a) \end{vmatrix} = \begin{vmatrix} 1 & \frac{2a}{a+1} & \frac{2a-1}{a} \\ 0 & \frac{2(b-a)}{(a+1)(b+1)} & \frac{2b-1}{b}-\frac{2a-1}{a} \\ 0 & \frac{2(c-a)}{(a+1)(b+1)} & \frac{2b-1}{b}-\frac{2a-1}{a} \end{vmatrix}$$

On Simplifying the determinant leads to

$$\frac{2\,(b-a)(c-a)(c-b)}{abc(a+1)(b+1)(c+1)} \geq 0$$

$$\frac{2\,(b-a)(c-a)(c-b)}{abc(a+1)(b+1)(c+1)} = \frac{2V(a;r=0,k=0)}{abc(a+1)(b+1)(c+1)} \geq 0 \tag{3.3}$$

Assume that $0 < a < b < c$ implies that $abc(a+1)(b+1)(c+1) \geq 0$

Similarly by considering $h(x) = \frac{2x-1}{x}$ and $k(x) = \frac{2x}{x+1}$, by lemma (2.3) we have

$$\frac{2\,(b-a)(c-a)(b-c)}{abc(a+1)(b+1)(c+1)} = \frac{2V(a;r=0,k=0)}{abc(a+1)(b+1)(c+1)} \leq 0 \tag{3.4}$$

By combining eqs (3.3) and (3.4) the proof of theorem (3.2) completes.

Theorem 3.3. The Geometric mean is concave (convex) with respect to $F_7(a,b)$ if and only if $V\left(a; r = -3/2, k = 0\right) \leq (\geq)0$.

Proof: Consider the Geometric mean and $F_7(a,b)$ in the form;

$$G(x,1) = \sqrt{x} \text{ and } F_7(x,1) = \frac{x^2-x+1}{x}$$

Let $h(x) = \sqrt{x}$ and $k(x) = \frac{x^2-x+1}{x}$, by lemma (2.3) we have

$$\begin{vmatrix} 1 & h(a) & k(a) \\ 0 & h(b)-h(a) & k(b)-k(a) \\ 0 & h(c)-h(a) & k(c)-k(a) \end{vmatrix} = \begin{vmatrix} 1 & \sqrt{a} & \frac{a^2-a+1}{a} \\ 0 & \sqrt{b}-\sqrt{a} & \frac{b^2-b+1}{b}-\frac{a^2-a+1}{a} \\ 0 & \sqrt{c}-\sqrt{a} & \frac{c^2-c+1}{c}-\frac{a^2-a+1}{a} \end{vmatrix}$$

On Simplifying the determinant leads to

$$\frac{(\sqrt{b}-\sqrt{a})(\sqrt{c}-\sqrt{a})(\sqrt{c}-\sqrt{b})}{abc}(abc + \sqrt{ac} + \sqrt{ab} + \sqrt{bc}) \geq 0$$

Then by lemma (2.2)

$$(3.5) \quad \frac{(\sqrt{b}-\sqrt{a})(\sqrt{c}-\sqrt{a})(\sqrt{c}-\sqrt{b})}{abc}(abc + \sqrt{ac} + \sqrt{ab} + \sqrt{bc})$$

$$= \frac{V(a;r=-3/2,k=0)}{abc}(abc + \sqrt{ac} + \sqrt{ab} + \sqrt{bc}) \geq 0$$

Similarly by considering $h(x) = \frac{x^2-x+1}{x}$ and $k(x) = \sqrt{x}$, by lemma (2.3) we have

$$\frac{(\sqrt{b}-\sqrt{a})(\sqrt{c}-\sqrt{a})(\sqrt{c}-\sqrt{b})}{abc}(abc + \sqrt{ac} + \sqrt{ab} + \sqrt{bc}) \leq 0 \tag{3.6}$$

By combining eqs (3.5) and (3.6) the proof of theorem (3.3) completes.

Theorem 3.4. The Geometric mean is concave (convex) with respect to $F_9(a, b)$ if and only if $V(a; r = -3/2, k = 0) \leq (\geq)0$.

Proof: Consider the Geometric mean and $F_9(a, b)$ in the form;

$$G(x, 1) = \sqrt{x} \quad \text{and} \quad F_9(x, 1) = \frac{2x-1}{x}$$

Let $h(x) = \sqrt{x}$ and $k(x) = \frac{2x-1}{x}$, by lemma (2.3) we have

$$\begin{vmatrix} 1 & h(a) & k(a) \\ 0 & h(b) - h(a) & k(b) - h(a) \\ 0 & h(c) - h(a) & k(c) - k(a) \end{vmatrix} = \begin{vmatrix} 1 & \sqrt{a} & \frac{2a-1}{a} \\ 0 & \sqrt{b} - \sqrt{a} & \frac{2b-1}{b} - \frac{2a-1}{a} \\ 0 & \sqrt{c} - \sqrt{a} & \frac{2c-1}{c} - \frac{2a-1}{a} \end{vmatrix}$$

On Simplifying the determinant leads to

$$\frac{(\sqrt{b}-\sqrt{a})(\sqrt{c}-\sqrt{b})(\sqrt{c}-\sqrt{b})}{abc}(\sqrt{ac} + \sqrt{ab} + \sqrt{bc}) \leq 0 \tag{3.7}$$

By lemma (2.2) we have

$$\frac{(\sqrt{b}-\sqrt{a})(\sqrt{c}-\sqrt{b})(\sqrt{b}-\sqrt{c})}{abc}(\sqrt{ac} + \sqrt{ab} + \sqrt{bc})$$

$$= \frac{(\sqrt{b}-\sqrt{a})(\sqrt{c}-\sqrt{b})(\sqrt{b}-\sqrt{c})}{abc}(\sqrt{ac} + \sqrt{ab} + \sqrt{bc}) \leq 0$$

Similarly by considering $h(x) = \frac{x^2-x+1}{x}$ and $k(x) = \sqrt{x}$, by lemma (2.3) we have

$$\frac{(\sqrt{b}-\sqrt{a})(\sqrt{c}-\sqrt{b})(\sqrt{b}-\sqrt{c})}{abc}(\sqrt{ac} + \sqrt{ab} + \sqrt{bc}) \geq 0 \tag{3.8}$$

By combining eqs (3.7) and (3.8) the proof of theorem (3.4) completes.

Theorem 3.5: The unnamed mean $F_7(a, b)$ is concave (convex) with respect to $F_9(a, b)$ if and only if $V(a; r = 0, k = 0) \leq (\geq)0$

Proof: Consider $F_7(a, b)$ and $F_9(a, b)$ in the form;

$$F_7(x, 1) = \frac{x^2-x+1}{x} \text{ and } F_9(a, b) = \frac{2x-1}{x}$$

Let $h(x) = \frac{x^2-x+1}{x}$ and $k(x) = \frac{2x-1}{x}$

By lemma 2.3, we have

$$\begin{vmatrix} 1 & h(a) & k(a) \\ 0 & h(b) - h(a) & k(b) - h(a) \\ 0 & h(c) - h(a) & k(c) - k(a) \end{vmatrix} = \begin{vmatrix} 1 & \frac{a^2-a+1}{a} & \frac{2a-1}{a} \\ 0 & \frac{b^2-b+1}{b} - \frac{a^2-a+1}{a} & \frac{2b-1}{b} - \frac{2a-1}{a} \\ 0 & \frac{c^2-c+1}{c} - \frac{a^2-a+1}{a} & \frac{2c-1}{c} - \frac{2a-1}{a} \end{vmatrix}$$

On simplifying the determinant leads to $\frac{(b-a)(c-a)(b-c)}{abc} \leq 0$ \hfill (3.9)

By lemma 2.1, we have

$$\frac{(b-a)(c-a)(b-c)}{abc} = \frac{V(a;r=0,k=0)}{abc} \leq 0$$

Similarly by considering $h(x) = \frac{2x-1}{x}$ and $k(x) = \frac{x^2-x+1}{x}$

By lemma 2.3, we have $\frac{(b-a)(c-a)(b-c)}{abc} \geq 0$ \hfill (3.10)

By combining eqs (3.9) and (3.10) the proof of theorem (3.5) completes.

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