ISSN- 2350-0530(O), ISSN- 2394-3629(P)
DOI: https://doi.org/10.29121/granthaalayah.v5.i11.2017.2321

INTERNATIONAL JOURNAL OF RESEARCH – GRANTHAALAYAH
A knowledge Repository

Science

ON LACUNARY ARITHMETIC STATISTICAL CONTINUITY FOR DOUBLE SEQUENCES

M. M. Karagama [*1], **F. B. Ladan** [2]
[*1, 2] Department of Mathematical Sciences, University of Maiduguri, Borno State, Nigeria

Abstract

In this article, we shall introduce the concept of lacunary arithmetic statistical continuity for double sequences and investigate some inclusion relations.

Keywords: Summability; Arithmetic Statistical Convergence; Lacunary Arithmetic Statistical Convergence; Lacunary Arithmetic Statistical Continuity; Double Sequences.

Cite This Article: M. M. Karagama, and F. B. Ladan. (2017). "ON LACUNARY ARITHMETIC STATISTICAL CONTINUITY FOR DOUBLE SEQUENCES." *International Journal of Research - Granthaalayah*, 5(11), 22-26. 10.29121/granthaalayah.v5.i11.2017.2321.

1. Introduction

The concept of statistical convergence was introduced by Fast [4] and it was further investigated from the sequence space point of view and linked with summability theory by Fridy [2], Connor [3], Fridy and Orhan [1], Šalát [5] and many others.

While the idea of arithmetic convergence was introduced by Ruckle [9]. Yaying and Hazarika [8] used this concept of arithmetic convergence and introduced arithmetic statistical convergence and lacunary arithmetic statistical convergence of single sequence. Also Yaying and Hazarika [8] establish some sequential properties of lacunary arithmetic statistical continuity of single sequence. The concept of statistical convergence of double sequences was introduced by Mursaleen [6]. Using the method of Mursaleen, we shall extend the results of Yaying and Hazarika [8] to double sequences as follows:

2. Lacunary Arithmetic Statistical Continuity *(First we Noted)*

Definition 2.1: (Yaying and Hazarika [2017]) A sequence $x = (x_k)$ is called arithmetically convergent if for each $\varepsilon > 0$ there is an integer l such that for every integer k we have $\left|x_k - x_{\langle k,l \rangle}\right| < \varepsilon$, where the symbol $\langle k,l \rangle$ denotes the greatest common divisor of two integers k and l. We denote the sequence space of all arithmetic convergent sequence by AC.

 ISSN- 2350-0530(O), ISSN- 2394-3629(P)
DOI: 10.5281/zenodo.1065890

Definition 2.2: (Fridy and Orhan [1993]) Let $\theta = (k_r)$ be a lacunary sequence. A number sequence $x = (x_k)$ is said to be lacunary statistically convergent to l or S_θ-convergent to l, if, for each $\varepsilon > 0$,

$\lim_{r\to\infty} \frac{1}{h_r} \ |\{k \in \ I_r : |x_k - l| \geq \varepsilon\}| = 0$

In this case, one writes $S_\theta - \lim x_k \ = l$ or $x_k \rightarrow (S_\theta)$.The set of all lacunary statistically convergence sequences is denoted by S_θ

Definition 2.3: (Yaying and Hazarika [2017]) A sequence $x = (x_k)$ is said to be arithmetic statistically convergent if for each $\varepsilon > 0$, there is an integer *l* such that

$$\lim_{n\to\infty} \frac{1}{n} \left|\left\{k \in \ n: \left|x_k - \ x_{\langle k,l \rangle}\right| \ \geq \ \varepsilon \ \right\}\right| = 0$$

We shall use ASC to denote the set of all arithmetic statistical convergent sequences. Thus for $\varepsilon > 0$ and integer l

$$\text{ASC} = \left\{(x_k) : \lim_{n\to\infty} \frac{1}{n} \left|\left\{k \in \ n: \left|x_k - \ x_{\langle k,l \rangle}\right| \ \geq \ \varepsilon \ \right\}\right| = 0 \ \right\}.$$

We shall write $ASC - \lim x_k \ = \ x_{\langle k,l \rangle}$ to denote the sequence (x_k) is arithmetic statistically convergent to$x_{\langle k,l \rangle}$.

Definition 2.4: (Yaying and Hazarika [2017]) Let $\theta = (k_r)$ be a lacunary sequence. The number sequence $x = (x_k)$ is said to be lacunary arithmetic statistically convergent if for each $\varepsilon > 0$ there is an integer *l* such that

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\left\{k \in \ I_r: \left|x_k - \ x_{\langle k,l \rangle}\right| \ \geq \ \varepsilon \ \right\}\right| = 0$$

We shall write

$$ASC_\theta \ = \left\{x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \left|\left\{k \in \ I_r: \left|x_k - \ x_{\langle k,l \rangle}\right| \ \geq \ \varepsilon \ \right\}\right| = 0 \ \right\}.$$

We shall write $ASC_\theta \ - \lim x_k \ = \ x_{\langle k,l \rangle}$ to denote the sequence (x_k) is lacunary arithmetic statistically convergent to$x_{\langle k,l \rangle}$.

Definition 2.5: (Yaying and Hazarika [2017]) A function f defined on a subset $E \ of \ \mathbb{R}$ is said to be lacunary arithmetic statistical continuous if it preserves lacunary arithmetic statistical convergence i.e. if

$$ASC_\theta \ - \lim x_k \ = \ x_{\langle k,l \rangle} \text{ Implies } ASC_\theta \ - \lim f(x_k) = f(x_{\langle k,l \rangle}).$$

Theorem 2.1: (Yaying and Hazarika [2017]) Let (f_m), $m \in \mathbb{N}$ be sequence of ASC_θ continuous functions defined on a subset of E of $\mathbb{R}$ and f_m , be uniformly convergent to a function f, then f is ASC_θ continuous.

 ISSN- 2350-0530(O), ISSN- 2394-3629(P)
DOI: 10.5281/zenodo.1065890

Theorem 2.2: (Yaying and Hazarika [2017]) The set of all ASC_θ continuous functions defined are on a subset E of $\mathbb{R}$ is a closed subset of all continuous function on E, i.e. $\overline{ASC_\theta(E)} = ASC_\theta(E)$, where $ASC_\theta(E)$ denotes the set of all ASC_θ continuous functions defined on E and $\overline{ASC_\theta(E)}$ denotes the closure of $ASC_\theta(E)$.

We shall now use the concept of statistical convergence to extend above concept and result to double sequences, using Analogy;

3. Lacunary Arithmetic Statistical Continuity For Double Sequences

Definition 3.1: A function f defined on a subset $D\ of\ \mathbb{R}$ is said to be lacunary arithmetic statistical continuous for double sequences if it preserves lacunary arithmetic statistical convergence for double sequences i.e. if

$$ASC_{\theta_{r,s}} - \lim x_{k,m} = x_{\langle\langle k,l\rangle,\langle m,n\rangle\rangle} \text{ Implies} ASC_{\theta_{r,s}} - \lim f(x_{k,m}) = f(x_{\langle\langle k,l\rangle,\langle m,n\rangle\rangle}).$$

Where the symbol $\langle k,l,m,n\rangle$ denotes the greatest common divisor of four integers k , l, m and n. We shall write $ASC_{\theta_{r,s}}$ continuous function to denote lacunary arithmetic statistical continuous for double sequences. It is easy to see that the sum and the difference of two $ASC_{\theta_{r,s}}$ continuous functions is $ASC_{\theta_{r,s}}$ continuous. Also the composition of two $ASC_{\theta_{r,s}}$ continuous functions is again $ASC_{\theta_{r,s}}$ continuous. In the classical case, it is known that the uniform limit of sequentially continuous function is sequentially continuous, now we see that the uniform limit of $ASC_{\theta_{r,s}}$ continuous functions is also $ASC_{\theta_{r,s}}$ continuous.

Theorem 3.1: Let $(f_{k,m})$, $k,m \in \mathbb{N}$ be sequence of $ASC_{\theta_{r,s}}$ continuous functions defined on a subset of D of $\mathbb{R}$ and $f_{k,m}$, be uniformly convergent to a function f, then f is $ASC_{\theta_{r,s}}$ continuous.

Proof 3.1: Let $\varepsilon > 0$ and $(x_{k,m})$ be any $ASC_{\theta_{r,s}}$ convergent sequence on a subset D of $\mathbb{R}$. By the uniform convergence of $f_{k,m}$, there exist $N \in \mathbb{N}$ such that $\left|f_{k,m}(x) - f(x)\right| \le \frac{\varepsilon}{3}$ for all $k,m \in N$ and for all$x \in D$. Since f_N is continuous on D, we have for an integerl,n.

$$\lim_{r,s\to\infty} \frac{1}{h_{r,s}} \left|\left\{k,m \in I_{r,s}: \left|f_N(x_{k,m}) - f_N(x_{\langle\langle k,l\rangle,\langle m,n\rangle\rangle})\right| \ge \frac{\varepsilon}{3}\right\}\right| = 0$$

On the other hand, for an integer l,n we have

$$\left\{k,m \in I_{r,s}: \left|f(x_{k,m}) - f(x_{\langle\langle k,l\rangle,\langle m,n\rangle\rangle})\right| \ge \frac{\varepsilon}{3}\right\} \subset \left\{k,m \in I_{r,s}: \left|f_N(x_{k,m}) - f(x_{\langle\langle k,l\rangle,\langle m,n\rangle\rangle})\right| \ge \frac{\varepsilon}{3}\right\}$$
$$\cup \left\{k,m \in I_{r,s}: \left|f_N(x_{\langle\langle k,l\rangle,\langle m,n\rangle\rangle}) - f_N(x_{k,m})\right| \ge \frac{\varepsilon}{3}\right\} \cup \left\{k,m \in I_{r,s}: \left|f_N(x_{k,m}) - f(x_{k,m})\right| \ge \frac{\varepsilon}{3}\right\}$$

 ISSN- 2350-0530(O), ISSN- 2394-3629(P)
DOI: 10.5281/zenodo.1065890

Thus it follows from the above inclusion that

$$\lim_{r,s\to\infty} \frac{1}{h_{r,s}} \left|\left\{k,m \in I_{r,s}: \left|f(x_{k,m}) - f(x_{\langle\langle k,l\rangle,\langle m,n\rangle\rangle})\right| \geq \varepsilon\right\}\right| \leq \lim_{r,s\to\infty} \frac{1}{h_{r,s}} \left|\left\{k,m \in I_{r,s}: \left|f_N(x_{\langle\langle k,l\rangle,\langle m,n\rangle\rangle}) - f(x_{\langle\langle k,l\rangle,\langle m,n\rangle\rangle})\right| \geq \frac{\varepsilon}{3}\right\}\right| + \lim_{r,s\to\infty} \frac{1}{h_{r,s}} \left|\left\{k,m \in I_{r,s}: \left|f_N(x_{\langle\langle k,l\rangle,\langle m,n\rangle\rangle}) - f_N(x_{k,m})\right| \geq \frac{\varepsilon}{3}\right\}\right| + \lim_{r,s\to\infty} \frac{1}{h_{r,s}} \left|\left\{k,m \in I_{r,s}: \left|f_N(x_{k,m}) - f(x_{k,m})\right| \geq \frac{\varepsilon}{3}\right\}\right|$$

Thus, f *is* $ASC_{\theta_{r,s}}$ continuous. ■

Theorem 3.2: The set of all $ASC_{\theta_{r,s}}$ continuous functions defined on a subset D of $\mathbb{R}$ is a closed subset of all continuous function on D, i.e. $\overline{ASC_{\theta_{r,s}}(D)} = ASC_{\theta_{r,s}}(D)$, where $ASC_{\theta_{r,s}}(D)$ denotes the set of all $ASC_{\theta_{r,s}}$ continuous functions defined on D and $\overline{ASC_{\theta_{r,s}}(D)}$ denotes the closure of $ASC_{\theta_{r,s}}(D)$.

Proof 3.2: Let f be any element of $\overline{ASC_{\theta_{r,s}}(D)}$ .Then there exist a sequence of points in $ASC_{\theta_{r,s}}(D)$ such that $lim f_{k,m} = f$. Now let $(x_{k,m})$ be any $ASC_{\theta_{r,s}}$ convergent sequence in D. Since $(f_{k,m})$ converges to f, there exist a positive integer *N* such that

$$\left|f(x) - f_{k,m}(x)\right| \leq \frac{\varepsilon}{3}, \forall\, k,m \geq N \text{ and } \forall\, x \in D$$

Now f_N is $ASC_{\theta_{r,s}}$ continuous on D, so we have for an integer l,n

$$\lim_{r,s\to\infty} \frac{1}{h_{r,s}} \left|\left\{k,m \in I_{r,s}: \left|f_N(x_{k,m}) - f_N(x_{\langle\langle k,l\rangle,\langle m,n\rangle\rangle})\right| \geq \frac{\varepsilon}{3}\right\}\right| = 0$$

On the other hand, for an integer l,n we have

$$\left\{k,m \in I_{r,s}: \left|f(x_{k,m}) - f(x_{\langle\langle k,l\rangle,\langle m,n\rangle\rangle})\right| \geq \frac{\varepsilon}{3}\right\}$$

$$\subset \left|\left\{k,m \in I_{r,s}: \left|f_N(x_{\langle\langle k,l\rangle,\langle m,n\rangle\rangle}) - f(x_{\langle\langle k,l\rangle,\langle m,n\rangle\rangle})\right| \geq \frac{\varepsilon}{3}\right\}\right|$$

$$\cup \left\{k,m \in I_{r,s}: \left|f_N(x_{\langle\langle k,l\rangle,\langle m,n\rangle\rangle}) - f_N(x_{k,m})\right| \geq \frac{\varepsilon}{3}\right\}$$

$$\cup \left\{k,m \in I_{r,s}: \left|f_N(x_{k,m}) - f(x_{k,m})\right| \geq \frac{\varepsilon}{3}\right\}$$

From the above inclusion we can write

$$\lim_{r,s\to\infty} \frac{1}{h_{r,s}} \left|\left\{k,m \in I_{r,s}: \left|f(x_{k,m}) - f(x_{\langle\langle k,l\rangle,\langle m,n\rangle\rangle})\right| \geq \varepsilon\right\}\right|$$

$$\leq \lim_{r,s\to\infty} \frac{1}{h_{r,s}} \left|\left\{k,m \in I_{r,s}: \left|f_N(x_{\langle\langle k,l\rangle,\langle m,n\rangle\rangle}) - f(x_{\langle\langle k,l\rangle,\langle m,n\rangle\rangle})\right| \geq \frac{\varepsilon}{3}\right\}\right|$$

$$+\ \lim_{r,s\to\infty}\frac{1}{h_{r,s}}\left|\left\{k,m\ \in\ I_{r,s}:\left|f_N\left(x_{\langle\,\langle k,l\rangle,\langle m,n\rangle\,\rangle}\right)-f_N\left(x_{k,m}\right)\right|\ \geq\ \frac{\varepsilon}{3}\right\}\right|$$

$$+\ \lim_{r,s\to\infty}\frac{1}{h_{r,s}}\left|\left\{k,m\ \in\ I_{r,s}:\left|f_N\left(x_{k,m}\right)-f\left(x_{k,m}\right)\right|\ \geq\ \frac{\varepsilon}{3}\right\}\right|=0$$

Thus f is $ASC_{\theta_{r,s}}$ continuous, so $f\ \in\ ASC_{\theta_{r,s}}(D)$ which gives us our required result. ■

References

[1] John Albert Fridy. (1993). Cihan Orhan, “LACUNARY STATISTICAL SUMMABILITY.” J. Math. Anal. Appl. 173(2) 497-504.

[2] John Albert Fridy. (1985). “ON STATISTICAL CONVERGENCE.” Analysis, 5, 301-313.

[3] J.S. Connor. (1988). “THE STATISTICAL AND STRONG P-CESARO CONVERGENCE OF SEQUENCES.” Analysis, 8, 47-63., ISSN (Online) 2196-6753, ISSN (Print) 0174-4747, DOI: https://doi.org/10.1524/anly.1988.8.12.47.

[4] H. Fast. (1951). “SUR LA CONVERGENCE STATISTIQUE.” Colloq Math, 2, 241-244.

[5] T. Šalát. (1980). “ON STATISTICALLY CONVERGENT SEQUENCES OF REAL NUMBERS.” Math Slovaca, 30, 139-15

[6] Mohammad Mursaleen, and O. H. H. Edely. (2003). “STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES.” J. Math.Anal.Appl., 288, 223-231.

[7] A. R. Freedman, J. J. Sember, M. Raphael. (1978). “SOME CESÁRO-TYPE SUMMABILITY SPACES.” Proc. Lond. Math. Soc. 37 508-520.

[8] Taja Yaying and Bipan Hazarika. (2017). “LACUNARY ARITHMETIC STATISTICAL CONVERGENCE.” arXiv:1703.03780v1 [Math.GM], 8 Mar.

[9] W. H. Ruckle. (2012). “ARITHMETICAL SUMMABILITY.” Jour. Math. Anal. Appl. *396* 741-748.

[10] John Albert Fridy, and Cihan Orhan. (1993). “LACUNARY STATISTICAL CONVERGENCE.” Pacific Journal of Mathematics, 160, no. 1, 43-53.

*Corresponding author.
E-mail address: mustaphakaragama @gmail.com