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ON SEPARATION AXIOMS IN TOPOLGICAL SPACES

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ABSTRACT

The purpose of this paper is to introduce weak separation axioms via sgp-closed sets in topological spaces and study some of their properties.

Keywords:
sgp-closed set, sgp-open set, sgp-T0, sgp-T1, sgp-T2.

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1. INTRODUCTION

General Topology plays very important role in all branches of Mathematics. An important concept in General Topology and Real Analysis concerns the variously modified forms of continuity and separation axioms etc. by utilizing the generalized closed sets.

In 1970, Levine [4] initiated the study of generalized closed(g-closed) sets, that is , a subset A of a topological space X is g-closed if the closure of A included in every open superset of A and defined a $T_{1/2}$ space to be one in which the closed sets and g-closed sets coincide. The notion has been studied extensively in recent years by many topologists. The study of g-closed sets has produced some new separation axioms. Some of these have been found useful in computer science and digital topology.

Recently Navalagi and Mahesh Bhat [7] introduced the notion of sgp-closed set utilizing pre closure operator. The notions of sgp-open sets, sgp-contonuity are introduced in [7]. In this paper we continue the study of sgp-closed sets, with introducing and characterizing weak forms of separation axioms.

2. PRELIMINARIES

Throughout this paper (X,τ) and (Y,σ)(or simply X and Y) denote topological spaces on which no separation axioms are assumed unless explicitly stated. If A is any subset of space X, then Cl(A) and Int(A) denote the closure of A and the interior of A in X respectively.
The following definitions are useful in the sequel:

Definition 2.1: A subset A of space X is called(i) a semi-open set [3] if $A \subseteq Cl(Int(A))$(ii) a semi-closed set [2] if $Int(Cl(A)) \subseteq A$ (iii) pre-open [5] , if $A \subseteq Int (Cl (A)))$. The complements of these sets are their respective closed sets in the space X.

Definition 2.3 [5]: The pre closure of a subset A of X is the intersection of all pre-closed sets containing A in X and is denoted by pcl(A).

Definition 2.2:A subset A of a space X is called

i. generalized-closed (in brief , g-closed) set [4] if $Cl (A) \subseteq U$ and U is open in X.The complement of g-closed set is g-open set.
ii. semi generalized pre closed (briefly, sgp-closed) set [7] if $pcl(A) \subseteq U$, whenever $A \subseteq U$ and U is semi-open in X.

The complement of sgp-closed set is sgp-open set and the family of all sgp-open sets of X is denoted by SGPO(X).

Definition 2.3[7]:A space X is said to be$_{sgp}T_c$-space ifevery sgp-closed set is closed set in it.

Definition 2.3[1]:A function f: $X \rightarrow Y$ is

i. sgp-irresolute if inverse image of sgp-closed set in Y is sgp-closed set in X.
ii. sgp-open if f(V) is sgp-open in Y for every open set V in X.

3. RESULTS AND DISCUSSIONS

We define and study the concept of sgp-T_0-space.

Definition 3.1: A topological space X is called sgp-T_0-space if for any pair of district points x, y of X, there exists sgp-open set G such that $x \in G$, $y \notin G$ or $x \notin G$, $y \in G$.

Example 3.2: Let X = {a, b} and $\tau = \{X, \phi, \{b\}\}$. Then (X, τ) is sgp-T_0-space, for distinct points a, b in X and {b} is the sgp-openset such that $a \notin \{b\}$, $b \in \{b\}$.

Theorem 3.3: Every subspace of a sgp-T_0-space is sgp-T_0-space.
Proof: Let X be a sgp-T_0-space and Y be a subspace of X. Let x and y be two distinct points of Y. As Y is a subspace of X, x, y are also distinct points of X. Since X is sgp-T_0-space, there exists a sgp-open set G such that $x \in G$, $x \notin G$. Then $Y \cap G$ is sgp-open in Y containing x but not y. Hence Y is sgp-T_0-space.

Theorem 3.4: If f: $X \rightarrow Y$ is injection sgp-irresolute function and Y is sgp-T_0-space, then X is sgp-T_0-space.

Proof: Suppose Y is sgp-T_0-space. Let a and b be two distinct points in X. Since f is an injection, f(a) and f(b) are distinct points in Y. Since Y is sgp-T_0-space, there exists sgp-open set G in Y such that f(a) $\in$ G and f(b) $\notin$ G. Again since f is sgp-irresolute, $f^{-1}(G)$ is sgp-open set in X such that a$\in f^{-1}(G)$ and b $\notin f^{-1}(G)$. Hence X is sgp-T_0-space.

Theorem 3.5: If X is sgp-T_0-space, ${}_{sgp}T_c$-space and Y is sgp-closed subspace of X, then Y is sgp-T_0-space.
Proof: Let X be sgp-T_0-space, ${}_{sgp}T_c$-space and Y is sgp-closed subspace of X. Let a and b be two distinct points of Y. As Y is subspace of X, a and b are two distinct points of X. Since X is sgp-T_0-space, there exists sgp-open set G such that a $\in$ G and b $\notin$ G. Again since X is ${}_{sgp}T_c$-space, G is open in X. Then Y $\cap$ G is open in Y. So a$\in$ Y $\cap$ G and b $\notin$ Y $\cap$ G. Hence Y is sgp-T_0-space.
Now, we introduce and studysgp-T_1-space

Definition 3.6: A topological space X is said to be sgp-T_1-space if for any pair of distinct points a and b there exist sgp-open sets G and H such that a $\in$ G, b $\notin$ G and a$\notin$ H, b $\in$ H.

Example 3.7: Let X = {a, b} and τ = {X,ϕ, {a}, {b}}. Then (X,τ) is a topological space. sgp-open sets are X, ϕ, {a}, {b}. Here a and b are two distinct points of X, then there exist sgp-open sets {a}, {b} of X such that a $\in$ {a}, a $\notin$ {b} and a $\notin$ {b}, b $\in$ {b}. Therefore X is sgp-T_1-space

Theorem 3.8: Every sgp-T_1-space is sgp-T_0-space but not conversely.
Proof: Let c and d be two distinct points of X. Since X is a sgp-T_1-space, there exist sgp-open sets G and H such that c $\in$ G, d $\notin$ G and c $\notin$H, d $\in$ H. We have c $\in$ G and d $\notin$ G. Therefore X is sgp-T_0-space

Example 3.9: Let X = {a, b} and τ = {X,ϕ, {b}}. Then X is a sgp-T_0-space but not a sgp-T_1-space. For any two distinct points a and b of X and {b} is sgp-open set such that a$\in$ {b}, b$\in$ {b}, but there is no sgp- open set G with a $\in$ G, b $\notin$ G for a $\neq$ b.

Theorem 3.10: If f: X $\rightarrow$ Y is a bijective sgp-open function. If X is a sgp-T_1-space and ${}_{sgp}T_c$-space, then Y is a sgp-T_1-space.
Proof: Let y_1 and y_2 be two distinct points of Y. Since f is bijective, there exist distinct points x_1 and x_2 of X such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since X is a sgp-T_1-space, there exist sgp-open sets G and H such that $x_1 \in G$ and $x_2 \notin G$ and $x_1 \notin H$ and $x_2 \in H$. Again since X is ${}_{sgp}T_c$-space, G and H are open sets in X. As f is sgp-open function, f(G) and f(H) are sgp-open sets such that $y_1 = f(x_1) \in f(G)$, $y_2 = f(x_2) \notin f(G)$ and $y_1 = f(x_1) \notin f(H)$, $y_2 = f(x_2) \in f(H)$. Hence Y is sgp-T_1-space.

Theorem 3.11: If X is sgp-T_1-space and ${}_{sgp}T_c$-space and Y is subspace of X, then Y is sgp-T_1-space.
Proof: Let X be a sgp-T_1-space and Y be a subspace of X. Let a and b be two distinct points of Y. Since X is a sgp-T_1-space, there exist sgp-open sets G and H such that a $\in$G, b $\notin$ G and a$\notin$ H, b $\in$ H. Again since X is a ${}_{sgp}T_c$-space, G and H are open sets in X. Then Y $\cap$ G and Y $\cap$ H are open sets so sgp-open sets of Y such that a $\in$ Y $\cap$ G, b $\notin$ Y $\cap$ G and a $\notin$Y $\cap$ H, b $\in$ Y $\cap$ H. Hence Y is sgp-T_1-space.

Theorem 3.12: If f: $X \rightarrow Y$ is injective sgp-irresolute function from a topological space X into sgp-T_1-space Y, then X is sgp-T_1-space.

Proof: Let a and b be two distinct points of X. Since f is injective, f(a) and f(b) are distinct points of Y. Since Y is sgp-T_1-space, there exist sgp-open sets G and H such that $f(a) \in G$, $f(b) \notin G$ and $f(a) \notin H$, $f(b) \in H$. Again since f is sgp-irresolute, $f^{-1}(G)$ and $f^{-1}(H)$ are sgp-open sets in X such that $a \in f^{-1}(G)$, $b \notin f^{-1}(G)$ and $a \notin f^{-1}(H)$, $b \in f^{-1}(H)$. Hence X is sgp-T_1-space.

Now, we definesgp-T_2-space.

Definition 3.13: A topological space X is said to be sgp-T_2-space if for any pair of distinct points a and b of X, there exist sgp-open sets x and y such that $a \in x$, $b \in y$ and $x \cap y = \phi$.

Example 3.14: Let X = {a, b} and $\tau = \{\phi, \{a\}, \{b\}, X\}$. Then (X,τ) is a topological space. sgp-open sets are X, ϕ, {a}, {b}. Here a and b are two distinct points of X, then their existsgp-open sets {a}, {b} of X such that $a \in \{a\}$, $b \in \{b\}$ and $\{a\} \cap \{b\} = \phi$. Therefore X is sgp-T_2-space.

Theorem 3.15: Every sgp-T_2-space is sgp-T_1-space.

Proof: Let X be a sgp-T_2-space. Let x and y be two distinct points of X. As X is sgp-T_2-space, there exist sgp-open sets G and H such that $x \in U$ and $y \in V$. This implies, $x \in U$, $y \notin U$ and $x \notin V$, $y \in V$. Hence X is sgp-T_1-space.

Theorem 3.16: If X is sgp-T_2-space, ${}_{sgp}T_c$-space and Y is a subspace of X, and then Y is also sgp-T_2-space.

Proof: Let X be a sgp-T_2-space and let Y be a subspace of X. Let x, y be two distinct points of Y. Since $Y \subseteq X$, x, y are distinct points of X. Again since X is sgp-T_2-space, there exist disjoint sgp-open sets G and H of x and y respectively. As X is ${}_{sgp}T_c$-space, there exist disjoints sgp-open sets G and H are open sets. So $G \cap Y$ and $H \cap Y$ are open set and so sgp-open set in Y. And also $x \in G$, $x \in Y$ implies $x \in G \cap Y$ and $y \in H$ and $y \in Y$ which implies $y \in Y \cap H$. Since $G \cap H = \phi$ we have $(Y \cap G) \cap (Y \cap H) = \phi$. Thus $G \cap Y$ and $H \cap Y$ are disjoint sgp-open sets of x and y respectively. Hence Y is sgp-T_2-space.

Theorem 3.17: If f: $X \rightarrow Y$ is a bijective sgp-open function. If X is sgp-T_2-space and ${}_{sgp}Tc$-space, then Y is also sgp-T_2-space.

Proof: The proof follows from the Theorem 3.16.

Theorem 3.18: Let X be a topological space. Then X is sgp-T_2-space if and only if the intersection of all sgp-closed neighborhood of each point of X is singleton.

Proof: Suppose X is sgp-T_2-space. Let x and y be any two distinct points of X. Since X is sgp-T_2-space, there exist open sets G and H such that $x \in G$, $y \in H$ and $G \cap H = \phi$. Since $G \cap H = \phi$ implies $x \in G \subseteq X - H$. So $X - H$ is sgp-closed neighbourhood of x, which does not contain y. Thus y does not belong to the intersection of all sgp-closed neighbourhood of x. Since y is arbitrary, the intersection of all sgp-closed neighbourhoods of x is the singleton {x}.

Conversely, let {x) be the intersection of all sgp-closed neighbourhoods of an arbitrary point $x \in X$. Let y be any point of X different from x. Since y does not belong to the intersection, there exists a sgp-closed neighbourhood N of x such that $y \notin N$. Since N is sgp-neighbourhood of x, there exists a sgp-open set G such $x \in G \subseteq X$. Thus G and X – N are sgp-open sets such that $x \in G$, $y \in X - N$ and $G \cap (X - N) = \phi$. Hence (X. τ) is sgp-T_2-space.

Theorem 3.19: Let (X, τ) be a topological space and let (Y, σ) be a sgp-T_2-space. Let f: (X, τ) → (Y, σ) be an injective sgp-irresolute map. Then (X, τ) is sgp-T_2-space.
Proof: Let x_1 and x_2 be any two distinct points of X. Since f is injective, $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$. Let $y_1 = f(x_1)$, $y_2 = f(x_2)$ so that $x_1 = f^{-1}(y_1)$, $x_2 = f^{-1}(y_2)$. Then $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Since (Y, μ) is sgp-T_2-space, there exist sgp-open sets G and H such that $y_1 \in G$, $y_2 \in G$ and $G \cap H = \phi$. As f is sgp-irresolute $f^{-1}(G)$ and $f^{-1}(H)$ are sgp-open sets of (X, τ). Now $f^{-1}(G) \cap f^{-1}(H) = f^{-1}(G \cap H) = f^{-1}(\phi) = \phi$ and $y_1 \in G$ implies $f^{-1}(y_1) \in f^{-1}(G)$ implies $x_1 \in f^{-1}(G)$, $y_2 \in H$ implies $f^{-1}(y_2) \in f^{-1}(H)$ implies $x_2 \in f^{-1}(H)$.Thus for every pair of distinct points x_1, x_2 of X there exist disjoint sgp-open sets $f^{-1}(G)$ and $f^{-1}(H)$ such that $x_1 \in f^{-1}(G)$, $x_2 \in f^{-1}(H)$. Hence (X, τ) is sgp-T_2-space.

4. NEW SEPARATION AXIOMS VIA sgp-OPEN SETS

Definition 4.1: Let X be a space. A subset $A \subset X$ is called a sgp-Difference set (in short sgp-D-set) if there are two sgp-open sets U, V in X such that $U \neq X$ and $A = U \setminus V$.
It is true that every sgp-open set $U \neq X$ is a sgp-D-set since $U = U \setminus \phi$.

Definition4.2: A space X is said to be
i. sgp-D_o if for x , y ∈ X containing x but not y or sgp-D-set containing y but not x.
ii. sgp-D_1 if for x , y ∈ X such that x ≠ y there exists a sgp-D-set of X containing x but not
iii. y and a sgp-D-set containing y but not x.
iv. sgp-D_2 if for x, y ∈X such that x ≠ y there exists a disjoint sgp-D- sets G and E such
v. that x∈ G and y ∈ E.

Theorem4.3: For a space X, the following properties hold:
i. If X is sgp-T_i , then it is sgp-T_{i-1} for i =1,2
ii. If X is sgp-T_i , then it is sgp-D_i for i = 0,1,2
iii. If X is sgp-D_i , then it is sgp-D_{i-1} for i =1,2
Proof: This is obvious from Definition 6.2

Theorem 4.4: For a space X, the following statements are true:
i. X is sgp-D_o if and only if X is sgp-T_o.
ii. sgp-D_1 if and only if X is sgp-D_2.
Proof: The sufficiency for (i) and (ii) follows from Theorem 4.3.

Necessity for (i). Let X be sgp-D_o so that for any pair of distinct points x and y of X at least one belongs to a sgp-D-set O. Therefore, we choose $y \in O$ and $y \notin O$. Suppose $O = U \setminus V$ for $U \neq X$ and sgp-open sets U and V. This implies that $x \in U$. For the case that $y \notin O$ we have (i) $y \notin U$, (ii) $y \in U$ and $y \in V$. For (i) the space X is sgp-T_o since $x \in U$ and $y \notin U$.
For (ii), the space X is also sgp-T_o since $y \in V$ but $x \notin V$.

Necessity for (ii): Suppose X is sgp-D_1. It follows from the definition that for any distinct points x and y in X there exists sgp-D-sets G and E such that G containing x but not y and E containing y but not x .Let G = U \ V and E = W \ D, where U, V, W and D aresgp-open. By the fact that x $\notin$ E, we have two cases, i.e. either x $\notin$ W or both W and D contain x. If x $\notin$ W, then from y $\notin$ G either (i) y $\notin$ U or (ii) y $\in$ U and y $\in$ V.
If (i) is the case, then it follows from x $\in$ U \ V that x $\in$ U \ (V $\cup$ W) and also it follows from y $\in$ W \ D that y $\in$ W \ (U $\cup$ D). Thus we have U\ (V $\cup$ W) and W \ (U $\cup$D) which are disjoint.
If (ii) is the case, it follows from that x $\in$ U \ V and y $\in$ V since y $\in$ U and y $\in$ V. Therefore (U\V) $\cap$ V = ϕ . If x$\in$ W and x $\in$ D, we have y $\in$ W \ D and x $\in$D. Hence (W \ D) $\cap$ D =ϕ. This shows that X is sgp-D_2.

Corollary 4.5: If X is sgp-D_1 , then it is sgp-T_o

Theorem 4.6: If f $\rightarrow$ Y is a sgp-irresolute surjective function and S is a sgp-D-set in Y, then f^{-1}(S)is a sgp-D-set in X.
Proof: Let S be a sgp-D-set in Y. Then there are sgp-open sets U and V in Y such that S = U \ V and U $\neq$ Y. By the sgp-irresolute of f, f^{-1}(U) and f^{-1}(V) are sgp-open sets in X. Since U $\neq$ Y, we have f^{-1}(U) $\neq$ X. Hence f^{-1}(S) = f^{-1}(U) \ f^{-1}(V) is a sgp-D-set in X.

Theorem 4.7: If Y is sgp-D_1 and f $\rightarrow$ Y is a sgp-irresolute and bijective function, then X is sgp-D_1.
Proof: Suppose that Y is sgp-D_1 space. Let x and y be any pair of distinct points in X. Since f is injective and Y is sgp-D_1, there exist sgp-D-sets S_x and S_y of S containing f (x) and f (y) respectively, such that f (y) $\notin S_y$. By the Theorem 4.6, $f^{-1}(S_x)$ and $f^{-1}(S_y)$ are sgp-D-sets in X containing x and y respectively. This implies that X is a sgp-D_1 space.

Theorem 4.8: A space X is sgp-D_1 if and only if for each pair of distinct points x and y in X, there exists a sgp-irresolute surjective function f from X onto sgp-D_1 space Y such that f (x) $\neq$ f (y)
Proof: Necessity: For every pair of distinct points of X it suffices to take the identity mapping on X.
Sufficiency: Let x and y be any pair of distinct points in X. By hypothesis, there exists a sgp-irresolute, surjective function f of a space X onto a sgp-D_1 space Y such that f (x) $\neq$ f (y). Therefore, there exist disjoint sgp-D-sets S_x and S_y in Y such that f (x) $\in S_x$ and f (y) $\in S_y$. Since f is sgp-irresolute and surjective, by Theorem 4.6, $f^{-1}(S_x)$ and $f^{-1}(S_y)$ are disjoint sgp-D-sets in X containing x and y respectively. Hence by Theorem 6.4 (ii), X is a sgp-D_1 space.

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