INTERNATIONAL JOURNAL OF RESEARCH – GRANTHAALAYAH
A knowledge Repository

Science

AN EXTRAPOLATORY QUADRATURE RULE FOR ANALYTIC FUNCTIONS

P. M. Mohanty [*1], M. Acharya [2]
[*1] S.C.S. College, Puri, INDIA
[2] ITER, SOA University, Bhubaneswar, INDIA

DOI: https://doi.org/10.29121/granthaalayah.v4.i6.2016.2631

ABSTRACT

A quadrature rule for the numerical evaluation of integrals of analytic functions along directed line segments in the complex plane has been formulated using the transformed rule based on Gauss Legendre two point quadrature formulas and an interpolatory three point rule. The degree of precision has been increased from five to seven.

Keywords:
Degree of precision, Taylors' series expansion, Gauss-Legendre rules.

AMS classification 65 D 30

Cite This Article: P. M. Mohanty, and M. Acharya, "AN EXTRAPOLATORY QUADRATURE RULE FOR ANALYTIC FUNCTIONS" International Journal of Research – Granthaalayah, Vol. 4, No. 6 (2016): 8-11.

1. INTRODUCTION

Integrals of analytic functions of a complex variable along directed line segments occur quite often in applied mathematics and physics. Birkhoff and Young [3] have formulated an illustrious five point degree five formula for the numerical approximation of the following integral:

$$I(f) = \int_L f(z)dz \qquad (1)$$

where f is an analytic function in the disk $\Omega = \{z: |z - z_0| \leq \rho, \rho > |h|\}$, L is a directed line segment from $z_0 - h$ to $z_0 + h$. Acharya and Nayak [1] and Acharya, Acharya and Pati [2] have modified the quadrature rule due to Birkhoff and Young [3] and have formulated quadrature rules of higher degree for the numerical evaluation of the integral $I(f)$. Lether [4] had also formulated the three point degree five transformed rule based on Gauss-Legendre quadrature for the numerical approximation of the integral $I(f)$ and has shown its superiority over the rule due to Birkhoff and Young [3]. Recently Milovanovic [5] has constructed the generalized quadrature rule of degree nine and more for the evaluation of the integral $I(f)$ from which the degree five, degree seven and other rules cited can be obtained as particular limiting cases.

The objective of the present paper is to construct a quadrature rule for the numerical approximation of the integral $I(f)$ .The rule derived is based on the 2-point transformed Gauss-Legendre rule and another 3-point interpolatory rule. Studying the leading terms in the expression for the errors and using the technique of extrapolation, the degree of precision of the rule has been raised from five to seven.

2. FORMULATION OF THE RULES

Lether [4] using the transformation $w = z_0 + ht, t \in [-1,1]$ has obtained the 3 point degree five rule for the numerical evaluation of the integral (f) . Proceeding in the same vein, the 2-point transformed rule can be obtained as

$$R_{2GL}(f) = h\{f\left(z_0 + \tfrac{h}{\sqrt{3}}\right) + f\left(z_0 - \tfrac{h}{\sqrt{3}}\right)\} \qquad (2)$$

The truncation error associated with the rules $R_{2GL}(f)$ is given by

$$E_{2GL}(f) = I(f) - R_{2GL}(f) \qquad (3)$$

As the function $f(z)$ is analytic in the domain Ω the Taylor series expansion of the function about the point z_0 is given by

$$f(z) = \sum_{n=0}^{\infty} a_n\,(z - z_0)^n \qquad (4)$$

where $a_n = f^n(z_0)/n!$ .Since the degree of precision of the rule $R_{2GL}(f)$ is three, using eqn.(4) in eqn.(3) we obtain after simplification

$$E_{2GL}(f) = \tfrac{8}{45}h^5 a_4 + \tfrac{40}{189}h^7 a_6 + O(h^9). \qquad (5)$$

we next consider the following interpolatory 3-point rule for the integral $I(f)$ with nodes at $z_0, z_0 + iph\,, z_0 - iph$ where $p \,\epsilon\, (0, 1]$ and $i = \sqrt{-1}$:

$$R(f) = Af''(z_0) + B\{f(z_0 + iph) + f(z_0 - iph)\} \qquad (6)$$

making the rule $R(f)$ exact for the monomials $1,(z - z_0)^2$ we get the coefficients A and B and the rule $R(f)$ as follows:

$$R(f) = (1 + p^2/3)h^3\, f''(z_0) + h\{f(z_0 + iph) + f(z_0 - iph)\} \qquad (7)$$

The truncation error associated with this rule can be obtained in the following form using Taylors' series expansion.

$$E(f) = (2/5 - 2p^4)h^5 a_4 + (2/7 + 2p^6)\,h^7 a_6 + O(h^9). \qquad (8)$$

Removing the leading terms involving a_4 in the expressions given by $E_{2GL}(f)$ and $E(f)$ (equations (5) and (8)), we obtain after simplification the following approximation for the integral $I(f)$:

$$Q_1(f,p) = \frac{4h^3}{15(3p^2-1)} f''(z_0) + \frac{4h}{5(9p^4-1)} \{f(z_0+iph) + f(z_0-iph)\}$$
$$+ \frac{9h(5p^4-1)}{5(9p^4-1)} \{f\left(z_0+\frac{h}{\sqrt{3}}\right) + f\left(z_0-\frac{h}{\sqrt{3}}\right)\} \qquad (9)$$

The truncation error associated with the rule $Q_1(f,p)$ which is given by $E_1(f,p) = I(f) - Q_1(f,p)$ can be prescribed in the following form:

$$E_1(f,p) = \frac{8}{105(9p^4-1)} (21p^6 + 25p^4 - 2)\, h^7 a_6$$
$$-\frac{4}{45}(p^4-1)\, h^9 a_8 + O(h^{11}). \qquad (10)$$

It is evident from eqn.(10) that the rule $Q_1(f,p)$ has degree of precision at least five for all values of p in $(0, \sqrt[4]{1/3}) \cup (\sqrt[4]{1/3}, 1]$. The rule $Q_1(f,p)$ has degree of precision seven if the coefficient of a_6 in eqn.(10) vanishes i.e. if p =0.506505044864556 (the real root of $21p^6 + 25p^4 - 2$=0). Denoting the rule $Q_1(f,p)$ as $\tilde{Q}(f)$ and its truncation error as $\tilde{E}(f)$ we have the following:

$$\tilde{E}(f) = -0.166\times h^9 a_8. \qquad (11)$$

3. CONCLUSION

Figure -1 shows the plot of the magnitude of the error coefficient (the leading term associated with a_6 in eqn. (10)) versus the parameter $p \in (0, \sqrt[4]{1/3}) \cup (\sqrt[4]{1/3}, 1]$. It is noted that the minimum of the coefficient is zero when p^* =0.506505044864556 .As a matter of fact for this value of p the rule has highest degree of precision. It is observed that when $p < \sqrt[4]{1/3}$, the rule is reasonably accurate.

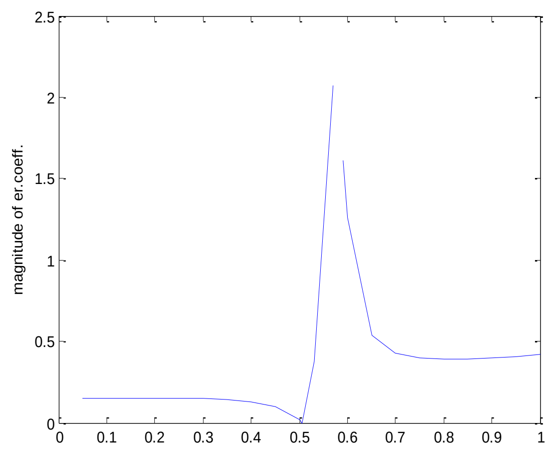

Figure 1:

ISSN- 2350-0530(O) ISSN- 2394-3629(P)
IF: 4.321 (CosmosImpactFactor), 2.532 (I2OR)

If we consider the integral $J=\int_L e^z dz$ where L is a directed line segment from *–i/2* to *i/2* and evaluate it by the rule $\tilde{Q}(f)$, then the approximate value obtained is correct upto eight decimal places with an error of the order of 10^{-9} for $p = p^*$.Therefore in addition to the simplicity of the rule it can be considered as an accurate one.

4. ACKNOWLEDGEMENTS

Authors are thankful to Professor B.P.Acharya for his guidance to complete this paper.

5. *REFERENCES*

[1] Acharya, B.P. and Nayak, P., "Numerical Evaluation of integrals of analytic functions", Int.J.Computer. Math., 59: 245-249, 1996.
[2] Acharya, M., Acharya, B.P. and Pati, S., "Numerical evaluation of integrals of analytic functions", Int. J. Comput.Math., 87(12):2747-2751, 2010.
[3] Birkhoff, G. and Young, D., "Numerical quadrature of analytic and harmonic functions", J.Math. Phy., 29: 217-221, 1950.
[4] Lether, F.G., "On Birkhoff-Young quadrature of analytic function", J.Comp. Appl.Math., 2: 81-84, 1976.
[5] Milovanovic, G.V.,"Generalised quadrature formulae for analytic functions", Appl. Math. Computations, 218: 8537-8551, 2012.