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APPLICATIONS OF EDGE COLORING OF GRAPHS WITH RAINBOW NUMBERS PHENOMENA

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ABSTRACT

This paper studies the Rainbow Ramsety Number for a non empty graph and the main results are 1. The Rainbow Ramsety Number of a graph F with out isolated vertices is defined if and only if F is a forest. 2. The Rainbow Ramsety Number of two graphs F1 and F2 with out isolated vertices is defined if and only if F1 is a star or F2 is a forest..

Mathematics Subject Classification 2000: 05CXX, 05C55, 05DXX, 05D10, 04XX, 04A10

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Rainbow Ramsety Number, forest, isolated vertices, star.

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1. INTRODUCTION

Basically in an edge-colored graph G that if there is a sub graph F of G all of whose edges are colored the same, then F is referred to as a monochromatic F. On the other hand, if all edges of F are colored differently, then F is referred to as a rainbow F.

2. DEFINITION

For a nonempty graph F, the **Rainbow Ramsety Number** RR (F) of F as the smallest positive integer n such that if each edge of the complete graph K_n is colored from any set of colors, then either a monochromatic F or a rainbow F is produced.

Let $\{v_1, v_2, \ldots V_n\}$ be the vertex set of a complete graph K_n. An edge coloring of K_n using positive integers for colors is called a **minimum coloring** if two edges v_iv_j and v_kv_l are colored the same if and only if
min {i, j} = {k, l}
while an edge coloring of K_n is called a **maximum coloring** if two edges u_iu_j and u_ku_l are colored the same if and only if
max {I,j} = max {k,l}

2.1. **Definion:** A graph with out cycles is a forest

2.2. **Theorem: Let F be a graph without isolated vertices. The Rainbow Ramsey number RR (F) is defined if and only if F is a forest.**

Let F be a graph of order $p \geq 2$. First we show that RR(F) is defined only if F is a forest. Suppose that F is not a forest. Thus F contains a cycle C, of length $k \geq 3$ say. Let n be an integer with $n \geq p$ and let $\{v_1,v_2,\ldots\ldots,v_n\}$ be the vertex set of a complete graph K_n. Define an edge coloring c of K_n by $c(v_iv_j) = i$ if $i < j$. Hence c is a minimum edge coloring of K_n. If k is the minimum positive integer such that v_k belongs to C, then two edges of C are colored k, implying that there is no rainbow F in K_n. Since any other edge in C is not colored k, it follows that F is not monochromatic either. Thus RR (F) is not defined.

For the converse, suppose that F is a forest of order $p \geq 2$. By known fact there exists and integer $n \geq p$ such that for any edge coloring of K_n with positive integers, there is a complete subgraph G of order p in K_n that is either monochromatic or rainbow or has minimum or maximum coloring. If G is monochromatic or rainbow, then K_n contains a monochromatic or rainbow F. Hence we may assume that the edge coloring of G is minimum or maximum, say the former. We show in this case that G contains a rainbow F. If F is not a tree, then we can add edges to F to produce a tree T of order p. Let
$V(G) = \{v_{i1},v_{i2},\ldots\ldots,v_{ip}),$
Where $i_1 < i_2 < \ldots. < i_p$. Select some vertex $v = v_{ip}$ of T and label the vertices of T in the order
$v = v_{ip}, v_{ip-1},\ldots.,v_{i2},v_{i1}$
of non decreasing distance from v; that is,
$d (v_{ij}, v) \geq d (v_{ij+1},v)$
for every integer j with $1 \leq j \leq p - 1$. Hence there exists exactly on edge of T having color i_j for each j with $1 \leq j \leq p -1$. Thus T and hence F is rainbow. The rainbow Ramsey number RR(F) is therefore defined.

2.3. **Example: For each integer $k \geq 2$, RR $(K_{1,k}) = (k - 1)^2 + 2$.**

Proof

We first show that $RR(K_{1,k}) \geq (k - 1)^2 + 2$. Let
$n = (k - 1)^2 + 1$.
We consider two cases.

Case 1. k is odd. Then n is odd Factor K_n into $^{n-1}/_2 = ^{(K-1)2}/_2$ Hamiltonian cycles each. Partition these cycles into k – 1 sets S_i $(1 \leq i \leq k-1)$ of $^{k-1}/_2$ Hamiltonian cycles each. Color each edge of each cycle in S_i with color i. then there is neither a monochromatic $K_{1,k}$ nor a rainbow $K_{1,k}$.

Case 2. k is even. Then n is even. Factor K_n into n-1 = $(k-1)^2$ 1-factors. Partition these 1-factors into k – 1 sets S_i $(1 \leq i \leq k-1)$ of k- 1 1-factors. Color each edge of each 1-factor in S_i color with i. Then there is neither a monochromatic $K_{1,k}$ nor a rainbow $K_{1,k}$.

Therefore, $RR(K_{1,k}) \geq (k-1)^2 + 2$. It remains to show that $RR(K_{1,k}) \leq (k-1)^2 + 2$. Let $N = (k-1)^2 + 2$ and let there be given an edge coloring of K_N from any set of colors. Suppose that no monochromatic $K_{1,k}$ results. Let v be a vertex of K_N. Since deg v = N – 1 and there is no monochromatic $K_{1,k}$, at most k -1 edges incident with v can be colored the same. Thus there are at least $[^N/_{k-1}] =$ k edges incident with v that are colored differently, producing a rainbow $K_{l,k}$.

More generally, for two nonempty graphs F_1 and F_2, the rainbow Ramsey number $RR(F_1,F_2)$ is defined as the smallest positive integer n such that if each edge of K_n is colored from any set of colors, then there is either a monochromatic F_1 or a rainbow F_2 defined for every pair F_1, F_2 of non empty graphs.

3. DEFINITION

If the partite sets u&w of a complete bi partite graph contain s&t vertices. Then this graph is denoted by $K_{s,t}$.the graph $K_{1,t}$ is called star.

***3.1.* Theorem: Let F_1 and F_2 be two graphs without isolated vertices. The rainbow Ramsey number RR (F_1,F_2) is defined if and only if F_1 is a star or F_2 is a forest.**

Proof. First, we show that RR (F_1,F_2) is exists only if F_1 is a star or F_2 is a forest. Suppose that F_1 is not a star and F_2 is not a forest. Let G be a complete graph of some order n such that V(G) = $\{v_1,v_2,\ldots,v_n\}$ and such that both F_1 and F_2 are sub graphs of G. Define an $(n-1)$ –edge coloring on G such that the edge v_iv_j is assigned the color i if $i < j$. Hence this coloring is a minimum edge coloring of G.

Let G_1 be any copy of F_1 in G and let a be the minimum integer such that v_a is a vertex of G_1. Then every edge incident with v_a is colored a. since G_1 is not a star, some edge of G_1 is not incident with v_a and is therefore not colored a. Hence G_1 is not monochromatic. Next, let G_2 be any copy of F_2 in G. Since G_2 is not a forest, G_2 contains a cycle C. Let b be the minimum integer such that v_b is a vertex of G_2 belonging to C. Since the two edges of C incident with v_b are colored b (and G_2 contains at least two edges colored b), G_2 is not a rainbow subgraph of G. Hence $RR(F_1,F_2)$ is not defined.

We now verify the converse. Let F_1 and F_2 be two graphs without isolated vertices such that either F_1 is a star or F_2 is a forest. We show that there exists a positive integer n such that for every edge coloring of K_n, either a monochromatic F_1 or a rainbow F_2 results. Suppose that the order of F_1 is s + 1 and the order of F_2 is t + 1for positive integers s and t. Hence $F_1 = K_{1,s}$. We now consider two cases, depending on whether F_1 is a star or F_2 is a forest. It is convenient to begin with the case where F_2 is a forest.

<u>Case 1.</u> F_2 is a forest. If F_2 is not a tree, then we may add edges to F_2 so that a tree G_2 results. If F_2 is a tree, then let $G_2 = F_2$. Furthermore, if F_1 is not complete, then we may add edges to F_1 so that a complete graph $G_1 = K_{s+1}$ results. If F_1 is complete, then let $G_1 = F_1$. Hence $G_1 = K_{s+1}$ and G_2 is a tree of order t + 1. We now show that RR (G_1,G_2) is defined by establishing the existence of a positive integer n such that any edge coloring of K_n from any set of colors results in either a monochromatic G_1 or a rainbow G_2. This, in turn, implies the existence of monochromatic F_1 or a rainbow F_2. We now consider two sub cases, depending on whether G_2 is a star.

Sub case 1.1. G_2 is a star of order t + 1, that is, $G_2 = K_{1,t}$. Therefore, in this subcase, $G_1 = K_{s+1}$ and $G_2 = K_{1,t}$. (This subcase will aid us later in the project) In this subcase, let

$$n = \sum_{i=0}^{(s-1)(t-1)+1} (t-1)^i$$

and let an edge coloring of K_n be given from any set of colors. If K_n contains a vertex incident with t or more edges assigned distinct colors, then K_n contains a rainbow G_2. Hence we may assume that every vertex of K_n is incident with at most t -1 edges assigned distinct colors. Let v_1 be a vertex of K_n. Since the degree of v_1 in K_n is n – 1, there are at least.

$$\frac{n-1}{t-1} = \sum_{i=0}^{(s-1)(t-1)} (t-1)^i$$

edges incident with u_1 that are assigned the same color, say color c_1.
Let S_1 be the set of vertices joined to v_1 by edges colored c_1 and let $v_2 \in S_1$.
There are at least

$$\frac{|S_1|-1}{t-1} \geq \sum_{i=0}^{(s-1)(t-1)-1} (t-1)^i$$

edges of the same color, say color c_2, joining v_2 and vertices of S_1, where possibly $c_2 = c_1$. Let S_2 be the set of vertices in S_1 joined to v_2 by edges colored c_2. Continuing in this manner, we construct sets $S_1,S_2,\ldots..S_{(s-1)\,(t-1)}$ and vertices,$v_1,v_2,\ldots.v_{(s-1)(t-1)+1}$ such that $2\leq i \leq (s-1)(t-1)+1$, the vertex v_i belongs to S_{i-1} and is joined to at least

$$\frac{|S_1|-1}{t-1} \geq \sum_{i=0}^{(s-1)(t-1)-1} (t-1)^i$$

vertices of S_{i-1} by edges colored c_i. Finally, in the set $S_{(s-1)\,(t-1)}$, the vertex

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$v_{(s-1)(t-1)+1}$ is joined to a vertex $v_{(s-1)(t-1)+2}$ in $S_{(s-1)(t-1)}$ by an edge colored $c_{(s-1)(t-1)+1}$. Thus we have a sequence

$$v_1, v_2, \ldots, v_{(s-1)(t-1)+2}$$

of vertices such that every edge $v_i v_j$ for $1 \leq i < j \leq (s-1)(t-1)+2$ is colored c_i and where the colors $c_1, c_2, \ldots c_{(s-1)(t-1)+1}$ are not necessarily distinct. In the complete subgraph H of order $(s-1)(t-1)+2$ induced by the vertices listed in (11.3), the vertex $v_{(s-1)(t-1)+2}$ is incident with at most $t-1$ edges having distinct colors. Hence there is a set of at least.

$$\left\lceil \frac{(s-1)(t-1)+1}{t-1} \right\rceil = s$$

Vertices in H joined to $v_{(s-1)(t-1)+2}$ by edges of the same color. Let $v_{i1}, v_{i2}, \ldots, v_{is}$ be s of these vertices, where $i_1 < i_2 < \ldots < i_s$. Then $c_{i1} = c_{i2} = \ldots = c_{is}$, and the complete subgraph of order s + 1 induced by
$\{v_{i1}, v_{i2}, \ldots, v_{is}, v_{(s-1)(t-1)+2}\}$
is monochromatic.

Subcase 1.2 G_2 is a tree of order t+1 that is not necessarily a star. Recall that $G_1 = K_{s+1}$. We proceed by induction on the positive integer t. If t =1 or t = 2, then G_2 is a star and the base case of the induction follows by subcase 1.1. Suppose that $RR(G_1,G_2)$ exists for $G_1 = K_{s+1}$ and for every tree G_2 of order t + 1where $t \geq 2$. Let T be a tree of order t + 2. We show that $RR(G_1,T)$ exists. Let v be an end-vertex of T and let v be the vertex of T that is adjacent to v. Let $T^l = T - v$. Since T^l is a tree of order t + 1, it follows by the induction hypothesis that $RR(G_1, T^l)$ exists, say $RR(G_1, T^l) = p$. Hence for any edge coloring of K_p from any set of colors, there is either a monochromatic $G_1 = K_{s+1}$ or a rainbow T^l. From sub case 1.1, we know that $RR(G_1, K_{1,t+1})$ exists. Suppose that $RR(G_1, K_{1,t+1}) = q$ and let n = pq in this subcase.

Let there be given an edge coloring of K_n using any number of colors. Consider a partition of the vertex set of K_n into q mutually disjoint sets of p vertices each. By the induction hypothesis, the complete subgraph induce by each set of p vertices contains either a monochromatic K_{s+1} or rainbow T^l. If a monochromatic K_{s+1} occurs in any of these complete subgraph K_p, then subcase 1.2 is verified. Hence we may assume that there are q pair wise mutually rainbow copies.

$T_1^l, T_2^l, \ldots, T_q^l$

of T^l, where u_i is the vertex in T_i^l ($1 \leq i \leq q$) corresponding to the vertex u in T^l.
Let H be the complete subgraph of order q induced by $\{u_1, u_2, \ldots, u_q\}$. Since $RR(K_{s+1}, K_{1,t+1}) = q$, it follows that either H contains a monochromatic K_{s+1} or a rainbow $K_{1,t+1}$. If H contains a monochromatic K_{s+1}, then once again, the proof of subcase 1.2 is complete. So we may assume that H contains a rainbow $K_{1,t+1}$. Let u_j be the center of a rainbow star $K_{1,t+1}$ in H. At least one of the t + 1 colors of the edges of $K_{1,t+1}$ is different from the colors of the t edges of T_j^l Adding the edge having this color at u_j in T_j^l produces a rainbow copy of T.

Case 2. F_1 is a star. Denote F_1 by G_1 as well and so $G_1 = K_{1,s}$. If F_2 is complete, then let $G_2 = F_2$. If F_2 is not complete, then we may add edges to F_2 so that a complete graph $G_2 = K_{t+1}$ results. We verify that RR (G_1, G_2) exists by establishing the existence of a positive integer n such that for any edge coloring of K_n from any set of colors, either a monochromatic G_1 or a rainbow G_2 results. This then shows that K_n will have a monochromatic F_1 or a rainbow F_2. For positive integers p and r with $r < p$, let

$$p^{(r)} = \frac{p!}{(p-r)!} = p(p-1)\cdots(p-r+1).$$

Now let n be an integer such that $s - 1$ divides n-1 and

$$n \geq 3 + \frac{(s-1)(t+2)^{(4)}}{8}$$

Then $n - 1 = (s-1)q$ for some positive integer q. Let there be given an edge coloring of K_n from any set of colors and suppose that no monochromatic $G_1 = K_{1,s}$ occurs. We show that there is a rainbow $G_2 = K_{t+1}$. Observe that the total number of different copies of K_{t+1} in K_n is $\binom{n}{t+1}$ implying the existence of at least one rainbow K_{t+1}.

First consider the number of copies of K_{t+1} containing adjacent edges uv and uw that are colored the same. There are n possible choice for the vertex u. suppose that there are a_i edges incident with u that are colored i for $1 \leq i \leq k$. Then

$$\sum_{i=1}^{k} a_i = n-1,$$

Where, by assumption, $1 \leq a_i \leq s - 1$ for each i. For each color $i (1 \leq i \leq k)$, the number of different choices for v and w where uv and uw are colored i is $\binom{a_i}{2}$. Hence the number of different choices for u and w where uv and uw are colored the same is

$$\sum_{i=1}^{k} \binom{a_i}{2}$$

since the maximum value of this sum occurs when each a_i is as large as possible, the largest value of this sum is when each a_i is $s - 1$ and when $k = q$, that is, there are at most

$$\sum_{i=1}^{q} \binom{s-1}{2} = q\binom{s-1}{2}$$

choices for v and w such that uv and uw are colored the same. Since there are $\binom{n-3}{t-2}$ choices for the remaining t – 2 vertices of K_{t+1}, it follows that there are at most

$$nq\binom{s-1}{2}\binom{n-3}{t-2}$$

copies of K_{t+1} containing two adjacent edges that are colored the same.
We now consider copies of K_{t+1} in which there two nonadjacent edges, say e = xy and f = wz, colored the same. There are $\binom{n}{2}$ choices for e and n – 2 choices for one vertex, say w, that is incident with f. The vertex w is incident with at most s – 1 edges having the same color as e and not adjacent to e. Since there are four ways of counting such a pair of edges in this way (namely e and either w or z, or f and either x or y), there are at most

$$\frac{\binom{n}{2}(n-2)(s-1)}{4} = \frac{n(n-1)(n-2)(s-1)}{8}$$

Ways to choose nonadjacent edges of the same color and $\binom{n-4}{t-3}$ ways to choose the remaining t - 3 vertices of K_{t+1}. Hence there are at most.

$$\frac{n(n-1)(n-2)(s-1)}{8}\binom{n-4}{t-3}$$

Copies of K_{t+1} containing two nonadjacent edges that are colored the same. Therefore, the number of non-rainbow copies of K_{t+1} is at most

$$\begin{aligned}
& nq\binom{s-1}{2}\binom{n-3}{t-2} + \frac{n(n-1)(n-2)(s-1)}{8}\binom{n-4}{t-3} \\
= \; & n\left(\frac{n-1}{s-1}\right)\frac{(s-1)(s-2)}{2}\left(\frac{n-2}{n-2}\right)\binom{n-3}{t-2} \\
& \quad + \frac{n(n-1)(n-2)(s-1)}{8}\left(\frac{n-3}{n-3}\right)\binom{n-4}{t-3} \\
= \; & \binom{n}{t+1}\left[\frac{(s-2)(t+1)^{(3)}}{2(n-2)} + \frac{(s-1)(t+1)^{(4)}}{8(n-3)}\right] \\
< \; & \binom{n}{t+1}\left[\frac{(s-1)(t+1)^{(3)}}{2(n-3)} + \frac{(s-1)(t+1)^{(4)}}{8(n-3)}\right] \\
= \; & \binom{n}{t+1}\left[\frac{(s-1)(t+1)^{(3)}(t+2)}{8(n-3)}\right] \\
= \; & \binom{n}{t+1}\left[\frac{(s-1)(t+2)^{(4)}}{8(n-3)}\right] \le \binom{n}{t+1},
\end{aligned}$$

Where the final inequality follows from known theorem, the rainbow Ramsey number is defined if and only if F is a forest hence there is a rainbow K_{t+1} in K_n.

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