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DEVELOPMENT OF ANDREWS-GARVAN-LIANG'S SELF-CONJUGATE S-PARTITIONS

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Abstract:
In 2013, Andrews, Garvan and Liang defined Self-conjugate S-partitions. In 2011, Andrews stated the definition of spt(n). This paper shows how to find the Self-conjugate S-partitions of 4 and 5 respectively, and proves the Corollary-1 that is 'The number of Self-conjugate S-partitions counted according to the weight w is congruent to the total number of smallest parts in all the partitions of n modulo 2'. This paper shows how to generate the generating functions for $M_{sc}(n)$ in different ways. This paper shows how to find the number of partitions of n with an odd number of smallest parts and a total number of even (respectively odd) parts, and shows how to find the number of partitions of n in odd parts without gaps, and also shows how to generate the generating functions for $\left(A_o(n)-A_e(n)\right)$ and $\left(L_1(n)-L_3(n)\right)$ respectively. This paper shows how to prove the further three Corollaries with the help of examples, and shows how to prove the three Theorems by easy algebraic method.

Keywords:
Crank, congruent to involution, product notations, Self-conjugate, spt(n), weight.

1. INTRODUCTION

We give some related definitions of P(n),Self-conjugate S-partitions, $M_{sc}(n)$, $A_e(n)$, $A_o(n)$, $L_1(n)$, $L_3(n)$, $(x)_\infty$, $(x;x^2)_\infty$, $(x^2;x^2)_\infty$, and spt(n). We give two tables of the Self-conjugate S-partitions of 4 and 5 respectively and introduce Corollary-1 in terms of $M_{sc}(n)$ and spt(n). We discuss the various generating functions for $M_{sc}(n)$, $\left(A_o(n)-A_e(n)\right)$ and $\left(L_1(n)-L_3(n)\right)$ and give some tables of the partitions of n with an odd number of smallest parts and of the partitions of n in odd parts without gaps. We discuss the number of Self-conjugate S-partitions counted according the weight *w*, and give further three Corollaries in terms of $\left(A_o(n)-A_e(n)\right)$, $\left(L_1(n)-L_3(n)\right)$ and $M_{sc}(n)$. Finally we prove the three Theorems with the help of various generating functions.

2. SOME RELATED DEFINITIONS

P(n)[7]:The number of partitions of n like 4, 3+1, 2+2, 2+1+1, 1+1+1+1 $\therefore$ P(4) = 5

Self-conjugate S-partitions [3,5]:

Let D denote the set of partitions into distinct parts and P denote the set of partitions. The set of vector partitions V is defined the Cartesian product

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$V = D \times P \times P$. If S is the subset of V,

$S = \{\vec{\pi} = (\pi_1, \pi_2, \pi_3) \in V : 1 \le (\pi_1) < \infty$ and $s(\pi_1) \le \min\{s(\pi_2), s(\pi_3)\}$. Here $s(\pi)$ is the smallest part in the partition with the convention that $s(\phi) = \infty$ for the empty partition. We call the vector partitions in S simply S-partitions for $\vec{\pi} = (\pi_1, \pi_2, \pi_3) \in s$, we define the weight $w(\vec{\pi}) = (-1)^{\#(\pi_1)-1}$, the Crank $\left(\vec{\pi}\right) = \#(\pi_2) - \#(\pi_3)$ and $|\vec{\pi}| = |\pi_1| + |\pi_2| + |\pi_3|$, When $|\pi_j|$ is the sum of the parts of π_j and # (π_j) denotes the number of parts of (π_j)

The map T: $S \to S$ given by,

$T(\vec{\pi}) = T(\pi_1, \pi_2, \pi_3) = T(\pi_1, \pi_3, \pi_2)$ is natural involution. An S-partition $\vec{\pi} = (\pi_1, \pi_3, \pi_2)$ is a fixed point of T if and only if $\pi_2 = \pi_3$. We call these fixed points "Self-conjugate S-partitions".
The number of Self-conjugate S-partitions counted according to the weight w is denoted by $M_{sc}(n)$, So that,

$$M_{sc}(n) = \sum_{\substack{\vec{\pi} \in S, |\vec{\pi}| = n \\ T(\vec{\pi}) = \vec{\pi}}} w(\vec{\pi})$$

$M_{sc}(n)$: The number of self-conjugate s-partitions counted according to the weight w is denoted by $M_{sc}(n)$, So that

$$M_{SC}(n) = \sum_{\substack{\vec{\pi} \in S, |\vec{\pi}| = n \\ T(\vec{\pi}) = \vec{\pi}}} w(\vec{\pi})$$

$A_e(n)$: The number of partitions of n with an odd number of smallest parts and a total number of even parts.

$A_o(n)$: The number of partitions of n with an odd number of smallest parts, and a total number of odd parts.

$L_1(n)$: The number of partitions of n in odd parts with no gaps, and the largest part is congruent to 1 mod 4.

$L_3(n)$: The number of partitions of n in odd parts with no gaps, and the largest part is congruent to 3 mod 4.

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Product notations [6]:

$(x)_\infty = (1-x)(1-x^2)(1-x^3)....$

$(x;x^2)_\infty = (1-x)(1-x^3)(1-x^5)....$

$(x^2;x^2)_\infty = (1-x^2)(1-x^4)(1-x^6)....$

$(-x;x)_\infty = (1+x)(1+x^2)(1+x^3)....$

spt(n) : spt(n) The total numbers of appearances of the smallest parts in all the partitions of n like,

n		spt(n)
1	$\dot{1}$	1
2	$\dot{2}$, $\dot{1}+\dot{1}$	3
3	$\dot{3}$, $2+\dot{1}$, $\dot{1}+\dot{1}+\dot{1}$	5
4	$\dot{4}$, $3+\dot{1}$, $\dot{2}+\dot{2}$, $2+\dot{1}+\dot{1}$, $\dot{1}+\dot{1}+\dot{1}+\dot{1}$	10

....

3. THERE ARE TWO TABLES OF THE SELF-CONJUGATE S-PARTITIONS

OF 4 AND 5: We get;

Table-1

Self-conjugate S-partition of 4	Weight $w(\vec{\pi})$	Crank $(\vec{\pi})$
$\vec{\pi}_1 = (4, \phi, \phi)$	+ 1	0
$\vec{\pi}_2 = (3+1, \phi, \phi)$	- 1	0
	$\sum w(\vec{\pi}) = 0$	

$\therefore$ $M_{sc}(4) = 0$. Here we have used ϕ to indicate the empty partition. Again;

Table-2

Self-conjugate S-partition of 5	Weight $w(\vec{\pi})$	Crank $(\vec{\pi})$
$\vec{\pi}_1 = (5, \phi, \phi)$	+ 1	0
$\vec{\pi}_2 = (1,2,2)$	+ 1	0
$\vec{\pi}_3 = (1,1+1,1+1)$	+ 1	0
$\vec{\pi}_4 = (1+4, \phi, \phi)$	- 1	0

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$\vec{\pi}_5 = (2+3, \phi, \phi)$	- 1	0
$\vec{\pi}_6 = (2+1, 1,1)$	-1	0
	$\sum w(\vec{\pi}) = 0$	

$\therefore$ $M_{sc}(5) = \sum_{i=1}^{6} w(\pi_i) = 1 + 1 + 1 - 1 - 1 - 1 = 0.$

Corollary-1 $M_{sc}(n) \equiv spt(n) (\bmod 2)$

Proof: From above table we get;

$$M_{sc}(1) = 1, M_{sc}(2) = 1, M_{sc}(3) = 1, ...$$

$$M_{sc}(3) = 1 \equiv 1(\bmod 2), spt(3) = 5 \equiv 1(\bmod 2)$$

$$M_{sc}(4) = 0 \equiv 0(\bmod 2), spt(4) = 10 \equiv 0(\bmod 2)$$

....

We can conclude that,

$M_{sc}(n) \equiv spt(n)(\bmod 2)$. Hence the Corollary.

4. NOW WE DESCRIBE THE GENERATING FUNCTIONS

The generation functions for $M_{sc}(n)$ are given by

$$\sum_{n=1}^{\infty} \frac{x^n (x^{n+1}; x)_\infty}{(x^{2n}; x^2)_\infty} = \frac{x(x^2; x)_\infty}{(x^2; x^2)_\infty} + \frac{x^2 (x^3; x)_\infty}{(x^4; x^2)_\infty} + \frac{x^3 (x^4; x)_\infty}{(x^6; x^2)_\infty}$$

$$= (x - x^4 - x^6 + ...) + (x^2 - x^5 - x^7 + ...) + (x^3 - x^7 + x^4 + x^5 + ...)$$

$$= (x + x^2 + x^3 + 0.x^4 + 0.x^5 + ...$$

$$= M_{sc}(1)x + M_{sc}(2)x^2 + M_{sc}(3)x^3 + M_{sc}(4)x^4 + M_{sc}(5)x^5 + ...$$

$$= \sum_{n=1}^{\infty} M_{sc}(n)x^n . \quad \therefore \sum_{n=1}^{\infty} M_{sc}(n)x^n = \sum_{n=1}^{\infty} \frac{x^n (x^{n+1}; x)_\infty}{(x^{2n}; x^2)_\infty} .$$

Again, we get;

$$\frac{1}{(-x ; x)_\infty} \sum_{n=1}^{\infty} \frac{x^n (-x; x)_{n-1}}{(1 - x^n)}$$

$$= \frac{1}{(1+x)(1+x^2)...} \left\{ \frac{x}{1-x} + \frac{x^2 (1+x)}{1-x^2} + \frac{x^3 (1+x)(1+x^2)}{1-x^3} + ... \right\}$$

$$= (1 - x - x^3 + x^4 - x^5 + ...)(x + x^3 + x^4 + x^5 + x^6 + ... + x^2 + x^3 + x^4 + ...)$$

$$= (1 - x - x^3 + x^4 - x^5)(x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + ...)$$

$$= x + x^2 + x^3 + 0.x^4 + 0.x^5 + ...$$

$$= M_{sc}(1)x + M_{sc}(2)x^2 + M_{sc}(3)x^3 + M_{sc}(4)x^4 + M_{sc}(5)x^5 + ...$$

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$$=\sum_{n=1}^{\infty} M_{sc}(n)x^n .$$

$$\therefore \sum_{n=1}^{\infty} M_{sc}(n)x^n = \frac{1}{(-x;x)_\infty} \sum_{n=1}^{\infty} \frac{x^n(-x;x)_{n-1}}{(1-x^n)} .$$

Again we get; $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{n^2}}{(x;x^2)_n} = \frac{x}{(1-x)} - \frac{x^4}{(1-x)(1-x^3)} + ...$

$$= x(1+x+x^2+x^3+x^4+x^5) - x^4(1+x+x^2+....)(1+x^3+....)$$

$$= x+x^2+x^3+0.x^4+0.x^5+......$$

$$= M_{sc}(1)x + M_{sc}(2)x^2 + M_{sc}(3)x^3 + M_{sc}(4)x^4 + M_{sc}(5)x^5 + ...)$$

$$= \sum_{n=1}^{\infty} M_{sc}(n)x^n . \quad \therefore \sum_{n=1}^{\infty} M_{sc}(n)x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{n^2}}{(x;x^2)_n}$$

Also we get; $\sum_{n=0}^{\infty} \frac{1}{(x^2;x^2)_n} \{(x)_{2n} - (x)_\infty\}$

$$= \{1-(1-x)(1-x^2).....\} + \left\{\frac{(1-x)(1-x^2)-(1-x)(1-x^2)......}{1-x^2}\right\}$$

$$+ \left\{\frac{(1-x)(1-x^2)(1-x^3)(1-x^4)-(1-x)(1-x^2)(1-x^3)....}{(1-x^2)(1-x^4)}\right\} + ...$$

$$= x+x^2+x^3+0.x^4+0.x^5+.......$$

$$= M_{sc}(1)x + M_{sc}(2)x^2 + M_{sc}(3)x^3 + M_{sc}(4)x^4 + M_{sc}(5)x^5 + ...)$$

$$= \sum_{n=1}^{\infty} M_{sc}(n)x^n . \quad \therefore \sum_{n=1}^{\infty} M_{sc}(n)x^n = \sum_{n=0}^{\infty} \frac{1}{(x^2;x^2)_n} \{(x)_{2n} - (x)_\infty\}.$$

The partitions of 5 with an odd number of smallest parts are in the table

Table-3

Partition (π) of 5	$\#(\pi)$
5	1
4+1	2
3+2	2
2+2+1	3
2+1+1+1	4
1+1+1+1+1	5

We see that $A_o(5)=3$, and $A_e(5)=3$. $\therefore$ $A_0(5)$ - $A_e(5)=0$.

The partitions of 6 with an odd number of smallest parts are in the table

Table-4

Partition (π) of 6	$\#(\pi)$

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6	1
5+1	2
4+2	2
3+2+1	3
3+1+1+1	4
2+2+2	3

We see that $A_0(6)=3$, $A_e(6)=3$. $\therefore$ $A_0(6) - A_e(6)=0$,
Similar, we get;
$A_0(1)- A_e(1)=1- 0=1$
$A_0(2)- A_e(2)=1- 0=1$
$A_0(3)- A_e(3)=1- 0=1$
$A_0(4)- A_e(4)=1-0=1$
....
The generating function for $(A_o(n)-A_e(n))$ is given by

$$\Sigma_{n=1}^{\infty} \frac{x^n}{(1-x^{2n})}\cdot\frac{1}{(-x^{n+1};x)_\infty}$$

$$=\frac{x}{(1-x^2)}\cdot\frac{1}{(1+x^2)(1+x^3)....}+\frac{x^2}{(1-x^4)(1+x^3)(1+x^4)...}+\frac{x^3}{(1-x^6)(1+x^4)(1+x^5)...}+.....$$

$$=x+x^2+x^3+0.x^4+0.x^5+0.x^6+-x^7+....$$

$$=\{A_o(1)-A_e(1)\}x+\{A_o(2)-A_e(2)\}x^2+\{A_o(3)-A_e(3)\}x^3+...$$

$$=\sum_{n=1}^{\infty}\{A_o(n)-A_e(n)\}x^n.$$

Corollary-2: $A_0(n)- A_e(n) = M_{sc}(n)$
Proof: We get, the generating function for $(A_0(n)- A_e(n))$ is

$$\Sigma_{n=1}^{\infty}\{A_0(n)-A_e(n)\}\ x^n=\Sigma_{n=1}^{\infty}\frac{x^n}{1-x^{2n}}\cdot\frac{1}{\left(-x^{n+1};x\right)_\infty}$$

$$=\frac{x}{(1-x)(1+x^2)(1+x^4)...}+\frac{x^2}{(1-x^4)(1+x^3)(1+x^4).....}+\frac{x^3}{(1-x^6)(1+x^4)(1+x^5)....}+....$$

$$=\frac{x}{(1-x)(1+x)(1+x^2)...}+\frac{x^2(1+x)}{(1-x^2)(1+x)(1+x^2).....}+\frac{x^3(1+x)(1+x^2)}{(1-x^3)(1+x)(1+x^2)....}+....$$

$$=\frac{1}{(1+x)(1+x^2)(1+x^3)...}\left\{\frac{x}{1-x}+\frac{x^2(1+x)}{1-x^2}+\frac{x^3(1+x)(1+x^2)}{1-x^3}+...\right\}$$

$$=\frac{1}{(-x;x)_\infty}\ \Sigma_{n=1}^{\infty}\frac{x^n(-x;x)_{n-1}}{1-x^n}=\Sigma_{n=1}^{\infty} M_{sc}(n)x^n$$

Equation the co-efficient of x^n from both sides we get;

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$A_0(n)$- $A_e(n)$= $M_{sc}(n)$. Hence the Corollary .
The partitions of 5 in odd parts with no gaps are in the table

Table-5

Partition (π) of 5	Largest part
3+1+1	3
1+1+1+1+1+	1

We see that $L_1(5)=1,\quad L_3(5)=1. \quad \therefore L_1(5)-L_3(5)=1-1=0$.
The partitions of 6 in odd parts with no gaps are in the table

Table-6

Partition (π) of 6	Largest part
3+1+1+1	3
1+1+1+1+1+1	1

We set that $L_1(6)=1,\quad L_3(6)=1. \quad \therefore L_1(6)-L_3(6)=1-1=0.$
Similarly we get;

$L_1(1)-L_3(1)=1-0=1$

$L_1(2)-L_3(2)=1-0=1$

$L_1(3)-L_3(3)=1-0=1$

$L_1(4)-L_3(4)=1-1=0$

....

The generating function for $(L_1(n)-L_3(n))$ is given by

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{n^2}}{(x;x^2)_n} = \frac{x}{1-x} - \frac{x^4}{(1-x)(1-x^3)} + \ldots\ldots$$

$$= x(1+x+x^2+x^3+x^4+x^5+\ldots.) - x^4(1+x+x^2+\ldots.)(1+x^3+\ldots.)+\ldots\ldots.$$

$$= x+x^2+x^3+0.x^4+0.x^5+0.x^6-x^7+\ldots\ldots$$

$$= \{L_1(1)-L_3(1)\}x+\{L_1(2)-L_3(2)\}x^2+\{L_1(3)-L_3(3)\}x^3+\ldots$$

$$= \sum_{n=1}^{\infty} L_1(n)-L_3(n)x^n .$$

Corollary-3 $A_0(n)$ - $A_e(n)$ = $L_1(n) - L_3(n) = M_{sc}(n)$

Proof: From above we get; $A_0(1)$ - $A_e(1)$ = 1 = $L_1(1) - L_3(1)$ and $M_{sc}(1) = 1$

$A_0(2)$ - $A_e(2)$ = 1 = $L_1(2) - L_3(2)$ and $M_{sc}(2) = 1$

$A_0(3)$ - $A_e(3)$ = 1 = $L_1(3) - L_3(3)$ and $M_{sc}(3) = 1$

....

We can conclude that,

$A_0(n)$ - $A_e(n)$ = $L_1(n) - L_3(n) = M_{sc}(n)$. Hence the Corollary .

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Corollary-4 $A_0(n) - A_e(n) = L_1(n) - L_3(n) \equiv spt(n) \pmod 2$

$A_0(1) - A_e(1) = L_1(1) - L_3(1) = 1 \equiv 1 \pmod 2$ and $spt(1) = 1 \equiv 1 \pmod 2$

$A_0(4) - A_e(4) = L_1(4) - L_3(4) = 0 \equiv 0 \pmod 2$ and $spt(4) = 10 \equiv 1 \pmod 2$

$A_0(6) - A_e(6) = L_1(6) - L_3(6) = 0 \equiv 0 \pmod 2$ and $spt(6) = 26 \equiv 0 \pmod 2$

....

We can conclude that,

$A_0(n) - A_e(n) = L_1(n) - L_3(n) \equiv spt(n) \pmod 2$. Hence the Corollary.

Theorem 1: $\sum_{n=1}^{\infty} \frac{x^n}{1-x^{2n}} \frac{1}{(-x^{n+1};x)_\infty} = \frac{1}{(-x;x)_\infty} \sum_{n=1}^{\infty} \frac{x^n(-x;x)_{n-1}}{(1-x^n)}$

Proof: Left hand side = $\sum_{n=1}^{\infty} \frac{x^n}{1-x^{2n}} \cdot \frac{1}{(-x^{n+1};x)_\infty}$

$$= \frac{x}{(1-x^2)} \cdot \frac{1}{(1+x^2)(1+x^3).....} + \frac{x^2}{(1-x^4)(1+x^3)(1+x^4)....} + \frac{x^3}{(1-x^6)(1+x^4)(1+x^5)....} + ..$$

$$= \frac{x}{(1-x)(1+x)(1+x^2)...} + \frac{x^2(1+x)}{(1-x^2)(1+x)(1+x^2).....} + \frac{x^3(1+x)(1+x^2)}{(1-x^3)(1+x)(1+x^2)....} + ...$$

$$= \frac{1}{(1+x)(1+x^2)(1+x^3)...} \left\{ \frac{x}{1-x} + \frac{x^2(1+x)}{1-x^2} + \frac{x^3(1+x)(1+x^2)}{1-x^3} + ... \right\}$$

$= \frac{1}{(-x;x)_\infty} \sum_{n=1}^{\infty} \frac{x^n(-x;x)_{n-1}}{1-x^n}$ = Right hand side. Hence the Theorem.

Theorem 2: $\frac{1}{(-x;x)_\infty} \sum_{n=1}^{\infty} \frac{x^n(-x;x)_{n-1}}{(1-x^n)} = \sum_{n=0}^{\infty} \frac{1}{(x^2;x^2)_n} \{(x)_{2n} - (x)_\infty\}$

Proof: We get; $\frac{1}{(x^2;x^2)_\infty} \sum_{n=1}^{\infty} \frac{x^n(-x;x)_{n-1}}{(1-x^n)}$

$$= \frac{1}{(1-x^2)(1-x^4)....} \left[\frac{x}{1-x} + \frac{x^2(1+x)}{1-x^2} + \frac{x^2(1+x)(1+x^2)}{1-x^3} + ... \right]$$

$$= \frac{1}{(1-x^2)(1-x^4)...} \left[\frac{x}{1-x} + \frac{x^2(1-x^2)}{(1-x)(1-x^2)} + \frac{x^3(1-x^2)(1-x^4)}{(1-x)(1-x^2)(1-x^3)} + ... \right]$$

$$= \frac{x}{(1-x)(1-x^2)(1-x^4)...} + \frac{x^2}{(1-x)(1-x^2)(1-x^4)....} + \frac{x^3}{(1-x)(1-x^2)(1-x^3)(1-x^6)....} + ...$$

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$$=\sum_{n=1}^{\infty}\frac{x^n}{(x)_n(x^{2n};x^2)_\infty}=\sum_{n=1}^{\infty}\frac{x^n}{(x)_n}\sum_{k=0}^{\infty}\frac{x^{2nk}}{(x^{2n};x^2)_k}[[1],P.19]$$

$$=\sum_{k=0}^{\infty}\frac{1}{(x^2;x^2)_k}\sum_{n=1}^{\infty}\frac{x^{n(2k+1)}}{(x)_n}=\sum_{k=0}^{\infty}\frac{1}{(x^2;x^2)_k}\left[\frac{x^{2k+1}}{(1-x)}+\frac{x^{2(2k+1)}}{(1-x)(1-x^2)}+....\right]$$

$$=\sum_{k=0}^{\infty}\frac{1}{(x^2;x^2)_k}\left[x^{2k+1}+x^{2k+2}+x^{2k+3}......\right]=\sum_{k=0}^{\infty}\frac{1}{(x^2;x^2)_k}\left[1+x^{2k+1}+x^{2k+2}+x^{2k+3}......-1\right]$$

$$=\sum_{k=0}^{\infty}\frac{1}{(x^2;x^2)_k}\left[\frac{1}{(1-x^{2k+1})(1-x^{2k+2})}......-1\right]$$

Multiplying both sides by $(x)_\infty$ We have;

$$\frac{(x)_\infty}{(x^2;x^2)_\infty}\sum_{n=1}^{\infty}\frac{x^n(-x;x)_{n-1}}{(1-x^n)}=\sum_{k=0}^{\infty}\frac{1}{(x^2;x^2)_k}\left[\frac{(x)_\infty}{(1-x^{2k+1})(1-x^{2k+2}).....}-(x)_\infty\right]$$

$$\frac{1}{(-x;x)_\infty}\sum_{n=1}^{\infty}\frac{x^n(-x;x)_{n-1}}{(1-x^n)}=\sum_{k=0}^{\infty}\frac{1}{(x^2;x^2)_k}\left[(x)_{2k}-(x)_\infty\right]$$

[Since

$$\frac{(x)_\infty}{(x^2;x^2)_\infty}=\frac{(1-x)(1-x^2)(1-x^3)(1-x^4).....}{(1-x^2)(1-x^4)(1-x^6).....}=\frac{1}{(1-x)(1-x^2)(1-x^3)....}=\frac{1}{(-x;x)_\infty}$$

And $\frac{(1-x)(1-x^2)...}{(1-x)(1-x^2)...}+\frac{(1-x)(1-x^2)(1-x^3)...}{(1-x^3)(1-x^4)....}+........$

$=1+(1-x)(1-x^2)+(1-x)(1-x^2)(1-x^3)(1-x^4)+......=\sum_{k=0}^{\infty}(x)_{2k}$]

$$\frac{1}{(-x;x)_\infty}\sum_{n=1}^{\infty}\frac{x^n(-x;x)_{n-1}}{(1-x^n)}=\sum_{n=0}^{\infty}\frac{1}{(x^2;x^2)_n}\{(x)_{2n}-(x)_\infty\}.$$

$\therefore$ Left hand side $=\sum_{n=0}^{\infty}\frac{1}{(x^2;x^2)_n}\{(x)_{2n}-(x)_\infty\}=$ Right hand side. Hence the Theorem.

Theorem 3: $\sum_{n=0}^{\infty}\frac{1}{(x^2;x^2)_n}\{(x)_{2n}-(x)_\infty\}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{n^2}}{(x;x^2)_n}$

Proof: Left hand side = $\sum_{n=0}^{\infty}\frac{1}{(x^2;x^2)_n}[(x;x)_{2n}-(x;x)_\infty]$

$$=\sum_{n=0}^{\infty}\left\{\frac{(x;x)_{2n}}{(x^2;x^2)_n}-\frac{(x;x)_\infty}{(x^2;x^2)_n}\right\}=\sum_{n=0}^{\infty}\left\{(x;x^2)_n-\frac{(x;x)_\infty}{(x^2;x^2)_n}\right\}$$

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$$\text{[Since } \frac{(1-x)(1-x^2)(1-x^3)(1-x^4).....}{(1-x^2)(1-x^4).....} = (1-x)(1-x^3).........] = \sum_{n=0}^{\infty}\left[(x\ ;x^2)_n-(x;x^2)_\infty+(x;x^2)_\infty-\frac{(x;x)_\infty}{(x^2;x^2)_n}\right]$$

$$=\sum_{n=0}^{\infty}\ [(x;x^2)_n-(x;x^2)_\infty]+\sum_{n=0}^{\infty}\ [(x;x^2)_\infty-\frac{(x;x)_\infty}{(x^2;x^2)_n}]$$

$$=-\sum_{n=1}^{\infty}\ \frac{(-1)^n x^{n^2}}{(x^2;x^2)_n}-(x;x^2)_\infty\sum_{n=1}^{\infty}\ \frac{x^{n^2}}{1-x^{2n}}+\sum_{n=0}^{\infty}\left\{(x;x^2)_\infty-\frac{(x)_\infty}{(x;x^2)_n}\right\}[4]$$

$$\text{with } x\to x^2,\ a\to o\ \text{ and } t=x$$

$$=-\sum_{n=1}^{\infty}\ \frac{(-1)^n x^{n^2}}{(x^2;x^2)_n}-(x;x^2)_\infty\sum_{n=1}^{\infty}\ \frac{x^{2n}}{1-x^{2n}}+\sum_{n=0}^{\infty}\left\{\frac{(x;x)_\infty}{(x^2;x^2)_\infty}-\frac{(x;x)_\infty}{(x^2;x^2)_n}\right\}$$

$$\textbf{[Since } (x;x^2)_\infty=(1-x)(1-x^3)...=\frac{(1-x)(1-x^2)...}{(1-x^2)(1-x^4)...}=\frac{(x;x)_\infty}{(x^2;x^2)_\infty}]$$

$$=-\sum_{n=1}^{\infty}\ \frac{(-1)^n x^{n^2}}{(x;x^2)_n}-\frac{(x;x)_\infty}{(x^2;x^2)_\infty}\sum_{n=1}^{\infty}\ \frac{x^{2n}}{1-x^{2n}}+(x;x)_\infty\sum_{n=0}^{\infty}\left\{\frac{1}{(x^2;x^2)_\infty}-\frac{1}{(x^2;x^2)_n}\right\}$$

$$=-\sum_{n=1}^{\infty}\ \frac{(-1)^n x^{n^2}}{(x;x^2)_n}-\frac{(x;x)_\infty}{(x^2;x^2)_\infty}\sum_{n=1}^{\infty}\ \frac{x^{2n}}{1-x^{2n}}+(x;x)_\infty\frac{1}{(x^2;x^2)_\infty}\sum_{n=1}^{\infty}\ \frac{x^{2n}}{1-x^{2n}}[4]$$

$$\text{with } x\to x^2,\ a=b=c=0$$

$$=-\sum_{n=1}^{\infty}\ \frac{(-1)^n x^{n^2}}{(x;x^2)_n}=\sum_{n=1}^{\infty}\ \frac{(-1)^{n-1}x^{n^2}}{(x;x^2)_n}=\text{Right hand side. Hence the Theorem .}$$

5. CONCLUSION

We have shown all Self-conjugate S-partitions of 4 and 5 respectively, and have satisfied the Corollary-1 $M_{sc}(n)\equiv spt(n)(\mathrm{mod}\,2)$ for any positive integral value of n. We have shown the partitions of n with an odd number of smallest parts for n = 5 and 6, and also have found the partitions of n in odd parts with no gaps for any positive integral value of n. We have introduced further three Corollaries in terms of $M_{sc}(n)$ and spt(n) respectively and have proved three Theorems of Self-conjugate S-partitions with the help of various generating functions.

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