DOI: https://doi.org/10.29121/granthaalayah.v2.i3.2014.3056

INTERNATIONAL JOURNAL of RESEARCH –GRANTHAALAYAH
A knowledge Repository
Science

"RAMANUJAN'S CONGRUENCES AND DYSON'S CRANK"

Sabuj Das[*1]
[*1] Senior Lecturer, Department of Mathematics, Raozan University College, BANGLADESH
*Correspondence Author: sabujdas.ctg@gmail.com

Abstract:
In 1944, Freeman Dyson conjectured the existence of a "crank" function for partitions that would provide a combinatorial result of Ramanujan's congruence modulo 11. In 1988, Andrews and Garvan stated such functions and described the celebrated result that the crank simultaneously explains the three Ramanujan congruences modulo 5, 7 and 11. Dyson wrote the article, titled Some Guesses in the theory of partitions, for Eureka, the undergraduate mathematics journal of Cambridge. He discovered the many conjectures in this article by attempting to find a combinatorial explanation of Ramanujan's famous congruences for P (n), the number of partitions of n indeed, Ramanujan's formulas lay unread until 1976 when Dyson found In the Trainty College Library of Cambridge University among papers from the estate of the late G.N.Watson. In 1986, F.Garvan wrote his Pennsylvania state Ph.D. Thesis Precisely on the formulas of Ramanujan relative to the crank. In view of this theoretical description, the story of the crank is a long romantic tale and the crank functions are intimately connected to all partitions congruences. In 2005, Mahlburg stated that the crank functions themselves obey Ramanujan type congruences.

Keywords:
Crank, congruences, Eureka, long romantic tale, modulo, theoretical, description.

1. INTRODUCTION

We give some related definitions of P(n) , crank of partitions, $(x)_\infty$, $(zx)_\infty$, $(x^2;x)_\infty$ and $M(m,n)$, and $M(m, t, n)$, π, $\#(\pi)$, $\sigma(\pi)$, crank of vector partitions, weight of $\vec{\pi}$, $M_V(m,n)$, $M_V(m,t,n)$ .This paper shows how to find the cranks of partitions of integers 7 and 8 by using tables 1 and 2 respectively and generate the generating functions for M(m, n) and M(o, n). This paper shows how to prove the Theorem 1 related the crank of partitions by taking individual function and describe the vector partitions of n. In this paper we prove the Mathematical results 1, 2, and 3 discovered by Dyson with the help of classification of the partitions of 9, 12, and 17 respectively and prove the Dyson's results with the help of examples. But these results 4, 5 and 6 are combinatorial results of Ramanujan's famous partition congruences modulo 5, 7 and 11 respectively. After the proof of Result-4 we show the relation between P (n) and $M_V(m,n)$ with the help of example.

1. Some related definitions:

$P(n)[Hardy, etel\,(1965)]$: Number of partitions of n, like 4, 3+1, 2+2, 2+1+1,1+1+1+1.

INTERNATIONAL JOURNAL of RESEARCH –GRANTHAALAYAH
A knowledge Repository
Science

Therefore, $P(4)=5$ and similarly, $P(5)=7$.

Crank of partitions [Andrews, etel (1988)]: For a partition, π, let $\lambda(\pi)$ denote the largest part of π, let $\mu(\pi)$ denote the number of ones in π, and let $\nu(\pi)$ denote the number of parts of π larger than $\mu(\pi)$. The crank $c(\pi)$ is given by;

$$c(\pi)=\begin{cases}\lambda(\pi), & \text{if } \mu(\pi)=0\\ \nu(\pi)-\mu(\pi), & \text{if } \mu(\pi)>0.\end{cases}$$

$$(x)_\infty=(1-x)(1-x^2)(1-x^3)\ldots$$

$$(zx)_\infty=(1-zx)(1-zx^2)(1-zx^3)\ldots$$

$$(x^2;x)_\infty=(1-x^2)(1-x^3)(1-x^4)\ldots$$

$M(m,n)$: The number of partitions of *n* with crank *m*.

$M(m,t,n)$: The numbers of partitions of *n* with crank congruent to *m* modulo *t* is denoted by $M(m,t,n)$.

π : A partition.

$\#(\pi)$: The number of parts of π.

$\sigma(\pi)$: The sum of the parts of π.

Crank of vector partitions: The number of parts of π_2 minus the number of parts of π_3, where π_2 and π_3 are unrestricted partitions in a vector partition $\vec{\pi}=(\pi_1,\pi_2,\pi_3)$ of *n*, if the sum of $\vec{\pi}$ is $s(\vec{\pi})=\sigma(\pi_1)+\sigma(\pi_2)+\sigma(\pi_3)=n$.

Weight of $\vec{\pi}$: Weight of vector partition $\vec{\pi}$ is defined as; $\omega(\vec{\pi})=(-1)^{\#(\pi_1)}$.

$M_V(m,n)$: The number of vector partitions of *n* (counted according to the weight ω) with crank *m*.

$M_V(m,t,n)$: The number of vector partitions of *n* (counted according to the weight ω) with crank congruent to *m* modulo *t*.

2. CRANKS OF PARTITIONS OF INTEGERS 7 AND 8:

Cranks of partitions of integers 7 and 8 are shown in the following Tables:

INTERNATIONAL JOURNAL of RESEARCH –GRANTHAALAYAH
A knowledge Repository
Science

Table 1: Cranks of partitions of integer 7.

Partitions π	Largest part $\lambda(\pi)$	Numbers of ones $\mu(\pi)$	Number of parts larger than $\mu(\pi)$ $\nu(\pi)$	Crank $c(\pi)$
7	7	0	1	7
6+1	6	1	1	0
5+2	5	0	2	5
5+1+1	5	2	1	–1
4+3	4	0	2	4
4+2+1	4	1	2	1
4+1+1+1	4	3	1	–2
3+3+1	3	1	2	1
3+2+1+1	3	2	1	–1
3+1+1+1+1	3	4	0	–4
3+2+2	3	0	3	3
2+1+1+1+1+1	2	5	0	–5
2+2+2+1	2	1	3	2
2+2+1+1+1	2	3	0	–3
1+1+1+1+1+1+1	1	7	0	–7
$P(7)=15$				$\sum c(\pi)=0$

Table2: Cranks of partitions of integer 8.

Partitions π	Largest part $\lambda(\pi)$	Numbers of ones $\mu(\pi)$	Number of parts larger than $\mu(\pi)$ $\nu(\pi)$	Crank $c(\pi)$
8	8	0	1	8
7+1	7	1	1	0
6+2	6	0	2	6
6+1+1	6	2	1	–1
5+3	5	0	2	5
5+2+1	5	1	2	1
5+1+1+1	5	3	1	–2
4+4	4	0	2	4
4+3+1	4	1	2	1
4+2+1+1	4	2	1	–1

INTERNATIONAL JOURNAL of RESEARCH –GRANTHAALAYAH
A knowledge Repository
Science

4+2+2	4	0	3	4
4+1+1+1+1	4	4	0	–4
3+3+2	3	0	3	3
3+3+1+1	3	2	2	0
3+2+1+1+1	3	3	0	–3
3+1+1+1+1+1	3	5	0	–5
3+2+2+1	3	1	3	2
2+2+2+2	2	0	4	2
2+2+2+1+1	2	2	0	–2
2+2+1+1+1+1	2	4	0	–4
2+1+1+1+1+1+1	2	6	0	–6
1+1+1+1+1+1+1+1	1	8	0	–8
$P(8)=22$				$\sum c(\pi)=0$

3. THE GENERATING FUNCTION FOR $M(m,n)$:

The generating function for $M(m,n)$ is given by [Andrews, etel (1988)];

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}M(m,n)z^m x^n=\prod_{n=1}^{\infty}\frac{(1-x^n)}{(1-zx^n)(1-z^{-1}x^n)}=\frac{(1-x)(1-x^2)(1-x^3)\dots}{(1-zx)(1-zx^2)\dots(1-z^{-1}x)(1-z^{-1}x^2)\dots}$$

$$=\frac{(1-x)(1-x^2)(1-x^3)\dots}{(1-zx)(1-zx^2)\dots(1-z^{-1}x)(1-z^{-1}x^2)\dots}=\frac{(1-x)}{(zx)_\infty}\left\{\frac{(1-x^2)(1-x^3)\dots}{(1-z^{-1}x)(1-z^{-1}x^2)\dots}\right\}$$

$$=\frac{(1-x)}{(zx)_\infty}\left\{\sum_{j=0}^{\infty}\frac{(zx)_j(xz^{-1})^j}{(x)_j}\right\}\quad\text{[Andrews (1985)],}$$

$$=\frac{(1-x)}{(zx)_\infty}\left\{1+\frac{(zx)_1(xz^{-1})^1}{(x)_1}+\frac{(zx)_2(xz^{-1})^2}{(x)_2}+\frac{(zx)_3(xz^{-1})^3}{(x)_3}+\dots\right\}=\frac{(1-x)}{(zx)_\infty}\left\{1+\frac{(1-zx)xz^{-1}}{(1-x)}+\frac{(1-zx)(1-zx^2)x^2z^{-2}}{(1-x)(1-x^2)}+\right\}$$

$$=\frac{(1-x)}{(1-zx)(1-zx^2)\dots}+\frac{xz^{-1}}{(1-zx^2)\dots}+\frac{x^2z^{-2}}{(1-x^2)(1-zx^3)\dots}+\frac{(1-zx)(1-zx^2)(1-zx^3)x^3z^{-3}}{(1-x)(1-x^2)(1-x^3)}+\dots$$

$$\frac{x^3z^{-3}}{(1-x^2)(1-x^3)(1-zx^4)\dots}+\dots$$

$$=\frac{(1-x)}{(1-zx)(1-zx^2)\dots}+\sum_{j=1}^{\infty}\frac{x^jz^{-j}}{(x^2;x)_{j-1}(zx^{j+1})_\infty}$$

$$=1+(z^{-1}+z-1)x+(z^{-2}+z^2)x^2+(z^{-3}+z^3+1)x^3+(1+z^{-2}+z^2+z^{-4}+z^4)x^4+$$

INTERNATIONAL JOURNAL of RESEARCH –GRANTHAALAYAH
A knowledge Repository
Science

$\left(1+z+z^{-1}+z^{3}+z^{-3}+z^{5}+z^{-5}\right)x^{5}+\left(1+z+z^{-1}+z^{2}+z^{-2}+z^{3}+z^{-3}+z^{4}+z^{-4}+z^{6}+z^{-6}\right)x^{6}+\ldots$

We see that the exponent of z represents the crank of partitions of n (for $n>1$). As for examples when n = 5 and 6. For n = 5:

Table-3: Cranks of partitions of integer 5.

Partitions of 5 (π)	Largest part $\lambda(\pi)$	Number of 1's $\mu(\pi)$	Number of parts larger than $\mu(\pi)$ $\nu(\pi)$	Crank $c(\pi)$
5	5	0	1	5
4+1	4	1	1	0
3+2	3	0	2	3
3+1+1	3	2	1	–1
2+2+1	2	1	2	1
2+1+1+1	2	3	0	–3
1+1+1+1+1	1	5	0	–5

For n = 6: **Table-4:** Cranks of partitions of integer 6.

Partitions of 6 π	Largest part $\lambda(\pi)$	Numbers of ones $\mu(\pi)$	Number of parts larger than $\mu(\pi)$ $\nu(\pi)$	Crank $c(\pi)$
6	6	0	1	6
5+1	5	1	1	0
4+2	4	0	2	4
4+1+1	4	2	1	–1
3+3	3	0	2	3
3+2+1	3	1	2	1
3+1+1+1	3	3	0	–3
2+2+2	2	0	3	2
2+2+1+1	2	2	0	–2
2+1+1+1+1	2	4	0	–4
1+1+1+1+1+1	1	6	0	–6

4. THE GENERATING FUNCTION FOR *M(0,n)* [Garvan (2013)].

The generating function for ***M(0,n)*** is defined as;

INTERNATIONAL JOURNAL of RESEARCH –GRANTHAALAYAH
A knowledge Repository
Science

$$(1-x)\sum_{n=0}^{\infty}\frac{x^{n(n+2)}}{(x^2)_n} = (1-x)\left[1+\frac{x^3}{(1-x)^2}+\frac{x^8}{(1-x)^2(1-x^2)^2}+\frac{x^{15}}{(1-x)^2(1-x^3)^2}+\dots\right]$$
$$=1-x+0.x^2+x^3+x^4+x^5+x^6+\dots$$
$$=M(0,0)+M(0,1)x+M(0,2)x^2+M(0,3)x^3+M(0,4)x^4+M(0,5)x^5+M(0,6)x^6+\dots$$
$$=\sum_{n=0}^{\infty}M(0,n)x^n .$$

Theorem-1: The number of partitions π of n with crank $c(\pi)=m$ is $M(m,n)$ for all $n>1$.

Proof: The generating function for $M(m,n)$ is given by;

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}M(m,n)z^m x^n = \prod_{n=1}^{\infty}\frac{(1-x^n)}{(1-zx^n)(1-z^{-1}x^n)} = \frac{(1-x)}{(1-zx)(1-zx^2)\dots}+\sum_{j=1}^{\infty}\frac{x^j z^{-j}}{(x^2;x)_{j-1}(zx^{j+1})_{\infty}} .$$

Now we distribute the function into two parts where first one represents the crank with $c(\pi)=\lambda(\pi)$ and second one represents the crank with $c(\pi)=\nu(\pi)-\mu(\pi)$.

The first function is;

$$\frac{(1-x)}{(1-zx)(1-zx^2)(1-zx^3)\dots}=1+(z-1)x+z^2x^2+z^3x^3+(z^2+z^4)x^4+(z^3+z^5)x^5+\dots$$

Counts (for $n>1$) the number of partitions with no 1's and the exponent on z being the largest part of the partition where $c(\pi)=\lambda(\pi)$, like; **Table-5:** Cranks of partitions of integer 4.

Partitions of 4 (π)	Largest part $\lambda(\pi)$	Number of 1's $\mu(\pi)$	Number of parts larger than $\mu(\pi)$ $\nu(\pi)$	Crank $c(\pi)$
4	4	0	1	4
2+2	2	0	2	2

Here n = 4, the 5th term is $(z^2+z^4)x^4$.

Again second function is,

$$\sum_{j=1}^{\infty}\frac{x^j z^{-j}}{(x^2,x)_{j-1}(zx^{j+1})_{\infty}} = \frac{xz^{-1}}{(1-zx^2)(1-zx^3)\dots}+\frac{x^2z^{-2}}{(1-x^2)(1-zx^3)(1-zx^4)\dots}+$$
$$\frac{x^3z^{-3}}{(1-x^2)(1-x^3)(1-zx^4)(1-zx^5)\dots}+\dots$$

INTERNATIONAL JOURNAL of RESEARCH –GRANTHAALAYAH
A knowledge Repository
Science

$= z^{-1}x + z^{-2}x^2 + (1+z^{-3})x^3 + (1+z^{-2}+z^{-4})x^4 + \ldots$

counts the number of partitions with $\mu(\pi) = j$ and the exponent on *z* is clearly $c(\pi) = \nu(\pi) - \mu(\pi)$, since $i > 0$, like; **Table-6:** Cranks of partitions of integer 4.

Partitions of 4 (π)	Largest part $\lambda(\pi)$	Number of 1's $\mu(\pi)$	Number of parts larger than $\mu(\pi)$ $\nu(\pi)$	Crank $c(\pi)$
3+1	3	1	1	0
2+1+1	2	2	0	–2
1+1+1+1	1	4	0	–4

Here *n* = 4, the 5th term is $(1+z^{-2}+z^{-4})x^4$ i.e., $(z^0+z^{-2}+z^{-4})x^4$.

Thus in the double series expansion of

$= \frac{(1-x)}{(1-zx)(1-zx^2)\ldots} + \sum_{j=1}^{\infty} \frac{x^j z^{-j}}{(x^2,x)_{j-1}(zx^{j+1})_{\infty}}$, we see that the coefficient of $z^m x^n$ $(n>1)$ is the number of partitions of *n* in which $c(\pi) = m$. Equating the coefficient of $z^m x^n$ from both sides in above function we get the number of partitions of *n* with $c(\pi) = m$ is $M(m,n)$ for all $n > 1$. Hence the Theorem .

5. MATHEMATICAL RESULTS

Notations:

The numbers of partitions of *n* with crank congruent to *m* modulo *t* is denoted by $M(m,t,n)$.

Some Dyson's Result:

Now we prove the following Dyson's result when *n* = 1 by computing;

Result -1: $M(0,5,5n+4) = M(1,5,5n+4) = M(2,5,5n+4) =$

$$M(3,5,5n+4) = M(4,5,5n+4) = \frac{P(5n+4)}{5},$$

Result -2 : $M(0,7,7n+5) = M(1,7,7n+5) = M(2,7,7n+5) = M(3,7,7n+5) =$

$$M(4,7,7n+5) = M(5,7,7n+5) = M(6,7,7n+5) = \frac{P(7n+5)}{7},$$

Result -3 : $M(0,11,11n+6) = M(1,11,11n+6) = M(2,11,11n+6) = M(3,11,11n+6) =$

$$M(4,11,11n+6) = M(5,11,11n+6) = \ldots = M(10,11,11n+6) = \frac{P(11n+6)}{11},$$ respectively.

INTERNATIONAL JOURNAL of RESEARCH –GRANTHAALAYAH
A knowledge Repository
Science

Proof of Result -1: The Classification of the Partitions of 9 on Cranks are given by Table-7.

Partitions with crank $\equiv 0 \pmod 5$	**Partitions with crank** $\equiv 1 \pmod 5$	**Partitions with crank** $\equiv 2 \pmod 5$	**Partitions with crank** $\equiv 3 \pmod 5$	**Partitions with crank** $\equiv 4 \pmod 5$
8+1 5+4 5+2+2 4+3+1+1 4+1+1+1+1+1 2+2+1+1+1+1+1	6+3 6+2+1 5+3+1 4+4+1 3+2+1+1+1+1 1+1+1+1+1+1+1+1+1	7+2 5+1+1+1+1 4+2+2+1 3+3+2+1 3+3+1+1+1 2+2+2+1+1+1	6+1+1+1 4+2+1+1+1 3+3+3 3+2+2+2 2+2+2+2+1 2+1+1+1+1+1+1+1	9 7+1+1 5+2+1+1 4+3+2 3+2+2+1+1 3+1+1+1+1+1+1
$M(0,5,9)=6$	$M(1,5,9)=6$	$M(2,5,9)=6$	$M(3,5,9)=6$	$M(4,5,9)=6$

But, $\frac{P(9)}{5} = \frac{30}{5} = 6$. Therefore,

$M(0,5,9) = M(1,5,9) = M(2,5,9) = M(3,5,9) = M(4,5,9) = \frac{P(9)}{5} = 6.$ Hence the Result .

Proof of Result -2: The Classification of the Partitions of 12 on Cranks are given by Table-8.

Partitions with crank $\equiv 0 \pmod 7$	Partitions with crank $\equiv 1 \pmod 7$	Partitions with crank $\equiv 2 \pmod 7$	Partitions with crank $\equiv 3 \pmod 7$	Partitions with crank $\equiv 4 \pmod 7$	Partitions with crank $\equiv 5 \pmod 7$	Partitions with crank $\equiv 6 \pmod 7$
5+3+2+1+1 7+3+2 7+5 . . . There are 11 partitions	6+5+1 8+3+1 8+2+2 . . . There are 11 partitions	6+3+2+1 7+2+2+1 2+2+2+2+2+2 . . . There are 11 partitions	7+1+1+1+1+1 3+3+3+3 3+3+2+2+2 . . . There are 11 partitions	6+2+1+1+1+1 4+4+4 4+4+2+2 . . . There are 11 partitions	7+1+1+1+1+1 3+3+3+3 3+3+2+2+2 . . . There are 11 partitions	6+2+1+1+1+1 4+4+4 4+4+2+2 . . . There are 11 partitions
$M(0,7,12)=11$	$M(1,7,12)=11$	$M(2,7,12)=11$	$M(3,7,12)=11$	$M(4,7,12)=11$	$M(5,7,12)=11$	$M(6,7,12)=11$

INTERNATIONAL JOURNAL of RESEARCH –GRANTHAALAYAH
A knowledge Repository
Science

But, $\frac{P(12)}{7}=\frac{77}{7}=11$ .Therefore, $M(0,7,12)=M(1,7,12)=M(2,7,12)=M(3,7,12)=M(4,7,12)=$ $M(5,7,12)=M(6,7,12)=\frac{P(12)}{6}$. Hence the Result .

Proof of Result -3: The Classification of the Partitions of 17 on Cranks are given by Table-9.

Partitions with crank $\equiv 0(\text{mod}11)$	Partitions with crank $\equiv 1(\text{mod}11)$	Partitions with crank $\equiv 2(\text{mod}11)$	Partitions with crank $\equiv 3(\text{mod}11)$	Partitions with crank $\equiv 4(\text{mod}11)$
16+1 11+6 11+4+2 . . . There are 27 partitions	8+8+1 12+5 12+3+2 . . . There are 27 partitions	6+5+5+1 13+4 13+2+2 . . . There are 27 partitions	4+4+4+4+1 14+3 3+3+3+3+3+2 . . . There are 27 partitions	15+2 4+4+4+4+3+2 4+4+3+3+3 . . . There are 27 partitions
$M(0,11,17)=27$	$M(1,11,17)=27$	$M(2,11,17)=27$	$M(3,11,17)=27$	$M(4,11,17)=27$

The Classification of the Partitions of 17 on Cranks are given by Table-10.

Partitions with crank $\equiv 5(\text{mod}11)$	Partitions with crank $\equiv 6(\text{mod}11)$	Partitions with crank $\equiv 7(\text{mod}11)$	Partitions with crank $\equiv 8(\text{mod}11)$	Partitions with crank $\equiv 9(\text{mod}11)$	Partitions with crank $\equiv 10(\text{mod}11)$
5+5+5+2 5+4+4+4 5+3+3+3+3 . . . There are 27 partitions	6+6+5 6+6+3+2 6+3+3+3+2 . . . There are 27 partitions	7+7+3 7+5+5 7+6+4 . . . There are 27 partitions	8+3+3+3 8+5+4 8+6+3 . . . There are 27 partitions	9+8 9+4+4 9+2+2+2+2 . . . There are 27 partitions	10+7 10+4+3 10+2+2+3 . . . There are 27 partitions
$M(5,11,17)=27$	$M(6,11,17)=27$	$M(7,11,17)=27$	$M(8,11,17)=27$	$M(9,11,17)=27$	$M(10,11,17)=27$

INTERNATIONAL JOURNAL of RESEARCH –GRANTHAALAYAH
A knowledge Repository
Science

But, $\frac{P(17)}{11}=\frac{297}{11}=27.$ Therefore $M(0,11,17)=M(1,11,17)=M(2,11,17)=M(3,11,17)=M(4,11,17)=M(5,11,17)=$

$$M(6,11,17)=M(7,11,17)=M(8,11,17)=M(9,11,17)=M(10,11,17)=\frac{P(17)}{11}=27.$$

Hence the Result.

6. THE CRANK FOR VECTOR PARTITIONS [Garvan (1988)].

For a partition π, let $\#(\pi)$ be the number of parts of π and $\sigma(\pi)$ be the sum of the parts of π with the convention $\#(\phi)=\sigma(\phi)=0$ for the empty partition ϕ of 0.

Let, $\vec{V}=\{(\pi_1,\pi_2,\pi_3)|\pi_1$ is a partition into unequal parts π_2, π_3 are unrestricted partitions}.

We shall call the elements of $\vec{V}$ vector partitions. For $\vec{\pi}=(\pi_1,\pi_2,\pi_3)$ in $\vec{V}$ we define the sum of parts, *s*, a weight, ω, and a crank, *c*, by;

$s(\vec{\pi})=\sigma(\pi_1)+\sigma(\pi_2)+\sigma(\pi_3)$.

$\omega(\vec{\pi})=(-1)^{\#(\pi_1)}$.

$c(\vec{\pi})=\#(\pi_1)-\#(\pi_2)$.

We say $\vec{\pi}$ is a vector partition of *n*, if $s(\vec{\pi})=n$. For example, if $\vec{\pi}=(1,1+1,1)$, then $s(\vec{\pi})=4$, $\omega(\vec{\pi})=-1$, $c(\vec{\pi})=1$ and $\vec{\pi}$ is a vector partition of 4.

The number of vector partitions of *n* (counted according to the weight ω) with crank *m* is denoted by $M_v(m,n)$ so that;

$M_V(m,n)=\sum\omega(\vec{\pi})$; if $\vec{\pi}\in\vec{V}$, $s(\vec{\pi})=n$, and $c(\vec{\pi})=m$

The number of vector partitions of *n* (counted according to the weight ω) with crank congruent to *k* modulo *t* is denoted by $M_V(k,t,n)$, so that;

$$M_V(k,t,n)=\sum_{m=-\infty}^{\infty}M_V(m,t+k,n)=\sum\omega(\vec{\pi});$$

if $\vec{\pi}\in\vec{V}$, $s(\vec{\pi})=n$, and $c(\vec{\pi})\equiv k(\bmod t)$.

Dyson's results on vector partitions:

The Result-4: $M_V(k,5,5n+4)=\frac{P(5n+4)}{5}$; $0\le k\le 4$.

Proof: We have 41 vector partitions of 4 are given in the following:

Table-11: The vector partitions of 4 with cranks

INTERNATIONAL JOURNAL of RESEARCH –GRANTHAALAYAH
A knowledge Repository
Science

Vector partitions of 4	Weight $\omega(\vec{\pi})$	Crank $(\vec{\pi})$
$\vec{\pi}_1 = (\phi, \phi, 4)$	+1	–1
$\vec{\pi}_2 = (\phi, \phi, 3+1)$	+1	–2
$\vec{\pi}_3 = (\phi, \phi, 2+2)$	+1	–2
$\vec{\pi}_4 = (\phi, \phi, 2+1+1)$	+1	–3
$\vec{\pi}_5 = (\phi, \phi, 1+1+1+1)$	+1	–4
$\vec{\pi}_6 = (\phi, 1, 3)$	+1	0
$\vec{\pi}_7 = (\phi, 1, 2+1)$	+1	–1
$\vec{\pi}_8 = (\phi, 1+1+1+1)$	+1	–2
$\vec{\pi}_9 = (\phi, 2+2)$	+1	0
$\vec{\pi}_{10} = (\phi, 2, 1+1)$	+1	–1
$\vec{\pi}_{11} = (\phi, 1+1, 2)$	+1	1
$\vec{\pi}_{12} = (\phi, 1+1, 1+1)$	+1	0
$\vec{\pi}_{13} = (\phi, 3, 1)$	+1	0
$\vec{\pi}_{14} = (\phi, 2+1, 1)$	+1	1
$\vec{\pi}_{15} = (\phi, 1+1+1, 1)$	+1	2
$\vec{\pi}_{16} = (\phi, 4, \phi)$	+1	1
$\vec{\pi}_{17} = (\phi, 3+1, \phi)$	+1	2
$\vec{\pi}_{18} = (\phi, 2+2, \phi)$	+1	2
$\vec{\pi}_{19} = (\phi, 2+1+1, \phi)$	+1	3
$\vec{\pi}_{20} = (\phi, 1+1+1+1, \phi)$	+1	4
$\vec{\pi}_{21} = (1, \phi, 3)$	–1	–1
$\vec{\pi}_{22} = (1, \phi, 2+1)$	–1	–2
$\vec{\pi}_{23} = (1, \phi, 1+1+1)$	–1	–3
$\vec{\pi}_{24} = (1, 1, 2)$	–1	0
$\vec{\pi}_{25} = (1, 1, 1+1)$	–1	–1
$\vec{\pi}_{26} = (1, 2, 1)$	–1	0
$\vec{\pi}_{27} = (1+1, 1, 1)$	–1	1

INTERNATIONAL JOURNAL of RESEARCH –GRANTHAALAYAH
A knowledge Repository
Science

$\vec{\pi}_{28} = (1,3,\phi)$	–1	1
$\vec{\pi}_{29} = (1,2+1,\phi)$	–1	2
$\vec{\pi}_{30} = (1,1+1+1,\phi)$	–1	3
$\vec{\pi}_{31} = (2,\phi,2)$	–1	–1
$\vec{\pi}_{32} = (2,\phi,1+1)$	–1	–2
$\vec{\pi}_{33} = (2,1,1)$	–1	0
$\vec{\pi}_{34} = (2,2,\phi)$	–1	1
$\vec{\pi}_{35} = (2,1+1,\phi)$	–1	2
$\vec{\pi}_{36} = (3,\phi,1)$	–1	–1
$\vec{\pi}_{37} = (2+1,\phi,1)$	+1	–1
$\vec{\pi}_{38} = (3,1,\phi)$	–1	1
$\vec{\pi}_{39} = (2+1,1,\phi)$	+1	1
$\vec{\pi}_{40} = (4,\phi,\phi)$	–1	0
$\vec{\pi}_{41} = (3+1,\phi,\phi)$	+1	0

From the above table we have;

$M_V(0,5,4) = \omega(\vec{\pi}_6) + \omega(\vec{\pi}_9) + \omega(\vec{\pi}_{12}) + \omega(\vec{\pi}_{13}) + \omega(\vec{\pi}_{24}) + \omega(\vec{\pi}_{26}) + \omega(\vec{\pi}_{33}) + \omega(\vec{\pi}_{40}) + \omega(\vec{\pi}_{41})$

= 1+1+1+1+1–1–1–1–1+1 = 1,

$M_V(1,5,4)$ = 1+1+1+1+1–1–1–1–1+1 = 1,

$M_V(2,5,4)$ = 1+1+1+1–1–1–1–1= 1,

$M_V(3,5,4)$ = 1+1+1–1–1+1–1= 1,

$M_V(4,5,4)$ = 1+1+1–1–1–1–1+1+1 = 1.

$\therefore\ M_V(0,5,4) = M_V(1,5,4) = M_V(2,5,4) = M_V(3,5,4) = M_V(4,5,4) = 1 = \frac{P(4)}{5}$, where $n = 0$.

In general we can conclude that;

$M_V(k,5,5n+4) = \frac{P(5n+4)}{5};\ 0 \le k \le 4$. Hence the Result .

Corollary-1: $P(n) = \sum_{m=-\infty}^{\infty} M_V(m,n)$

Proof: The generating function for $M_V(m,n)$ is;

Science
INTERNATIONAL JOURNAL of
RESEARCH –GRANTHAALAYAH
A knowledge Repository

$$\prod_{n=1}^{\infty}\frac{(1-x^n)}{(1-zx^n)(1-z^{-1}x^n)}=\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}M_V(m,n)z^n x^n$$

it was proved by Atkin and Swinnerton-Dyer (1954). By putting $z = 1$, we get;

$$\prod_{n=1}^{\infty}\frac{(1-x^n)}{(1-x^n)(1-x^n)}=\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}M_V(m,n)x^n$$

$$\Rightarrow\sum_{n=0}^{\infty}P(n)x^n=\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}M_V(m,n)x^n$$

Equating the coefficient of x^n from both sides we get;

$\therefore\ P(n)=\sum_{m=-\infty}^{\infty}M_V(m,n)$. Hence the Corollary .

Now we discuss it with an example;

R. H. S. = $\sum_{m=-\infty}^{\infty}M_V(m,n)=\sum_{m=-\infty}^{\infty}M_V(m,4)$, where n=4

$=\ldots+M_V(-4,4)+M_V(-3,4)+M_V(-2,4)+M_V(-1,4)+M_V(0,4)+M_V(1,4)+M_V(2,4)+M_V(3,4)+$
$M_V(4,4)+\ldots$

= 0 + 1 + 0 + 1 + 0 + 1 + 0 + 1 + 0 + 1= 5 = $P(4)$= L.H.S. $\therefore\ P(4)=\sum_{m=-\infty}^{\infty}M_V(m,4)$.

Again,

$M_v(0,4)=\omega(\vec{\pi}_6)+\omega(\vec{\pi}_9)+\omega(\vec{\pi}_{12})+\omega(\vec{\pi}_{13})+\omega(\vec{\pi}_{24})+\ \omega(\vec{\pi}_{26})+\omega(\vec{\pi}_{33})+\omega(\vec{\pi}_{40})+\omega(\vec{\pi}_{41})$

= 1+1+1+1–1–1–1–1+1 = 1

$M_v(1,4)=\omega(\vec{\pi}_{11})+\omega(\vec{\pi}_{14})+\ldots+\omega(\vec{\pi}_{39})$ = 1 + 1 + 1–1–1–1–1+1 = 0.

$M_v(-1,4)=\omega(\vec{\pi}_1)+\omega(\vec{\pi}_7)+\ldots+\omega(\vec{\pi}_{37})$= 1 + 1 + 1–1–1–1–1+1 = 0

$M_v(2,4)=$ 1 + 1 + 1–1–1= 1

$M_v(-2,4)=$ 1 + 1 + 1–1–1= 1

$M_v(3,4)=$ 1–1= 0

$M_v(-3,4)=$ 1–1= 0

$M_v(4,4)=$ 1

$M_v(-4,4)=$ 1

INTERNATIONAL JOURNAL of RESEARCH –GRANTHAALAYAH
A knowledge Repository
Science

$\sum M_{v}(m,4) = \sum \omega(\vec{\pi})$; i.e., $\sum_{m=-\infty}^{\infty} M_{v}(m,4) = \sum_{\substack{\vec{\pi}\in V \\ |\pi|=4 \\ \text{crank}(\vec{\pi})=m}} \omega(\vec{\pi}) = 5$

i.e., $\sum_{m=-\infty}^{\infty} M_{v}(m,4) = \sum_{\substack{\vec{\pi}\in V \\ |\pi|=4 \\ \text{crank}(\vec{\pi})=m}} \omega(\vec{\pi}) = P(4)$.

Corollary 2: $M_V(m,t,n) = M_V(t-m,t,n)$

By considering the transformation that interchanges π_2 and π_3 we have;
$M_V(m,n) = M_V(-m,n)$.
We illustrate with an example;
$M_V(1,4) = \omega(\vec{\pi}_{11}) + \omega(\vec{\pi}_{14}) + \ldots + \omega(\vec{\pi}_{39})$ = 1 + 1 + 1–1–1–1–1+1 = 0.
And $M_V(-1,4) = \omega(\vec{\pi}_1) + \omega(\vec{\pi}_7) + \ldots + \omega(\vec{\pi}_{37})$ = 1 + 1 + 1–1–1–1–1+1 = 0
$\therefore M_V(1,4) = M_V(-1,4)$.
Again,
$M_V(5-1,5,4) = M_V(4,5,4) = \omega(\vec{\pi}_{20}) = 1$
$\therefore M_V(1,5,4) = M_V(5-1,5,4)$ (By above)
Generally we can conclude that,
$M_V(m,t,n) = M_V(t-m,t,n)$.

The result-5: $M_V(k,7,7n+5) = \frac{P(7n+4)}{7}$; $0 \le k \le 6$.

Proof: We have the vector partitions of 5 are given in the following:

Table-12: The vector partitions of 5 with cranks

Vector partitions of 5	Weight $\omega(\vec{\pi})$	Crank $(\vec{\pi})$
$\vec{\pi}_1 = (\phi,\phi,5)$	+1	–1
$\vec{\pi}_2 = (\phi,\phi,4+1)$	+1	–2
$\vec{\pi}_3 = (\phi,\phi,3+2)$	+1	–2
$\vec{\pi}_4 = (\phi,\phi,3+1+1)$	+1	–3
$\vec{\pi}_5 = (\phi,\phi,2+2+1)$	+1	–3
$\vec{\pi}_6 = (\phi,\phi,2+1+1+1)$	+1	–4

INTERNATIONAL JOURNAL of RESEARCH –GRANTHAALAYAH
A knowledge Repository
Science

$\vec{\pi}_7 = (\phi, \phi, 1+1+1+1+1)$	+1	–5
$\vec{\pi}_8 = (5, \phi, \phi)$	–1	0
$\vec{\pi}_9 = (\phi, 5, \phi)$	+1	1
$\vec{\pi}_{10} = (\phi, 4+1, \phi)$	+1	2
$\vec{\pi}_{11} = (4+1, \phi, \phi)$	+1	0
$\vec{\pi}_{12} = (4, 1, \phi)$	–1	1
$\vec{\pi}_{13} = (1, 4, \phi)$	–1	1
$\vec{\pi}_{14} = (\phi, 4, 1)$	+1	0
$\vec{\pi}_{15} = (\phi, 1, 4)$	+1	0
$\vec{\pi}_{16} = (1, \phi, 4)$	–1	–1
$\vec{\pi}_{17} = (4, \phi, 1)$	–1	–1
$\vec{\pi}_{18} = (3+2, \phi, \phi)$	+1	0
$\vec{\pi}_{19} = (\phi, 3+2, \phi)$	+1	2
$\vec{\pi}_{20} = (3, 2, \phi)$	–1	1
$\vec{\pi}_{21} = (2, 3, \phi)$	–1	1
$\vec{\pi}_{22} = (\phi, 3, 2)$	+1	0
$\vec{\pi}_{23} = (\phi, 2, 3)$	+1	0
$\vec{\pi}_{24} = (3, \phi, 2)$	–1	–1
$\vec{\pi}_{25} = (2, \phi, 3)$	–1	–1
$\vec{\pi}_{26} = (\phi, 3+1+1, \phi)$	+1	3
$\vec{\pi}_{27} = (3+1, 1, \phi)$	+1	1
$\vec{\pi}_{28} = (1, 3+1, \phi)$	–1	2
$\vec{\pi}_{29} = (\phi, 3+1, 1)$	+1	1
$\vec{\pi}_{30} = (\phi, 1, 3+1)$	+1	–1
$\vec{\pi}_{31} = (3+1, \phi, 1)$	+1	–1
$\vec{\pi}_{32} = (1, \phi, 3+1)$	–1	–2
$\vec{\pi}_{33} = (3, 1+1, \phi)$	–1	2
$\vec{\pi}_{34} = (\phi, 1+1, 3)$	+1	1

INTERNATIONAL JOURNAL of RESEARCH –GRANTHAALAYAH
A knowledge Repository
Science

$\vec{\pi}_{35} = (\phi, 3, 1+1)$	+1	–1
$\vec{\pi}_{36} = (3, \phi, 1+1)$	–1	–2
$\vec{\pi}_{37} = (\phi, 2+2+1, \phi)$	+1	3
$\vec{\pi}_{38} = (1, 2+2, \phi)$	–1	2
$\vec{\pi}_{39} = (\phi, 2+2, 1)$	+1	1
$\vec{\pi}_{40} = (\phi, 1, 2+2)$	+1	–1
$\vec{\pi}_{41} = (1, \phi, 2+2)$	–1	–2
$\vec{\pi}_{42} = (2+1, 2, \phi)$	+1	1
$\vec{\pi}_{43} = (2, 2+1, \phi)$	–1	2
$\vec{\pi}_{44} = (\phi, 2, 2+1)$	+1	1
$\vec{\pi}_{45} = (\phi, 2+1, 2)$	+1	1
$\vec{\pi}_{46} = (2+1, \phi, 2)$	+1	–1
$\vec{\pi}_{47} = (2, \phi, 2+1)$	–1	–2
$\vec{\pi}_{48} = (\phi, 2+2+1, \phi)$	+1	4
$\vec{\pi}_{49} = (\phi, 2+1+1, 1)$	+1	2
$\vec{\pi}_{50} = (\phi, 1, 2+1+1)$	+1	–2
$\vec{\pi}_{51} = (1, 2+1+1, \phi)$	–1	3
$\vec{\pi}_{52} = (1, \phi, 2+1+1)$	–1	–3
$\vec{\pi}_{53} = (2+1, 1+1, \phi)$	+1	2
$\vec{\pi}_{54} = (\phi, 2+1, 1+1)$	+1	0
$\vec{\pi}_{55} = (\phi, 1+1, 2+1)$	+1	0
$\vec{\pi}_{56} = (2+1, \phi, 1+1)$	+1	–2
$\vec{\pi}_{57} = (\phi, 1+1+1, 2)$	+1	2
$\vec{\pi}_{58} = (\phi, 2, 1+1+1)$	+1	–2
$\vec{\pi}_{59} = (2, 1+1+1, \phi)$	–1	3
$\vec{\pi}_{60} = (2, \phi, 1+1+1)$	–1	–3
$\vec{\pi}_{61} = (\phi, 1+1+1+1+1, \phi)$	+1	5
$\vec{\pi}_{62} = (\phi, 1+1+1+1, 1)$	+1	3

INTERNATIONAL JOURNAL of RESEARCH –GRANTHAALAYAH
A knowledge Repository
Science

$\vec{\pi}_{63} = (\phi,1,1+1+1+1)$	+1	–3
$\vec{\pi}_{64} = (1,\phi,1+1+1+1)$	–1	–4
$\vec{\pi}_{65} = (1,1+1+1+1,\phi)$	–1	4
$\vec{\pi}_{66} = (\phi,1+1,1+1+1)$	+1	–1
$\vec{\pi}_{67} = (\phi,1+1+1,1+1)$	+1	1
$\vec{\pi}_{68} = (1,1,1+1+1)$	–1	–2
$\vec{\pi}_{69} = (1,1+1+1,1)$	–1	2
$\vec{\pi}_{70} = (1,1+1,1+1)$	–1	0
$\vec{\pi}_{71} = (1,1+1,2)$	–1	1
$\vec{\pi}_{72} = (1,2,1+1)$	–1	–1
$\vec{\pi}_{73} = (2,1+1,1)$	–1	1
$\vec{\pi}_{74} = (2,1,1+1)$	–1	–1
$\vec{\pi}_{75} = (2,2,1)$	–1	0
$\vec{\pi}_{76} = (2,1,2)$	–1	0
$\vec{\pi}_{77} = (1,2,2)$	–1	0
$\vec{\pi}_{78} = (3,1,1)$	–1	0
$\vec{\pi}_{79} = (1,3,1)$	–1	0
$\vec{\pi}_{80} = (1,1,3)$	–1	0
$\vec{\pi}_{81} = (1+2,1,1)$	+1	0
$\vec{\pi}_{82} = (1,1+2,1)$	–1	1
$\vec{\pi}_{83} = (1,1,1+2)$	–1	–1

From this table we have;

$M_V(0,7,5) = \omega(\vec{\pi}_8) + \omega(\vec{\pi}_{11}) + \omega(\vec{\pi}_{14}) + \omega(\vec{\pi}_{15}) + \omega(\vec{\pi}_{18}) + \omega(\vec{\pi}_{22}) + \omega(\vec{\pi}_{23}) + \omega(\vec{\pi}_{54}) + \omega(\vec{\pi}_{55}) + \omega(\vec{\pi}_{70}) + \omega(\vec{\pi}_{75}) + \omega(\vec{\pi}_{76}) + \omega(\vec{\pi}_{78}) + \omega(\vec{\pi}_{79}) + \omega(\vec{\pi}_{79}) + \omega(\vec{\pi}_{80}) + \omega(\vec{\pi}_{81})$

= –1+1+1+1+1+1+1+1+1–1–1–1–1–1–1–1–1+1 = 1.

Similarly, $M_V(0,7,5) = M_V(1,7,5) = \ldots = M_V(6,7,5) = 1 = \frac{P(5)}{7}$.

In general we can conclude that;

INTERNATIONAL JOURNAL of RESEARCH –GRANTHAALAYAH
A knowledge Repository
Science

$M_V(k,7,7n+5)=\frac{P(7n+5)}{7}$; $0\le k\le 6$. Hence the Result .

The Result-6: $M_V(k,11,11n+6)=\frac{P(11n+6)}{11}$; where, k=0 and n=0.

Proof: The vector partitions of 6 are given in the following:

Table-13: The vector partitions of 6 with cranks

Vector partitions of 6	Weight $\omega(\vec{\pi})$	Crank $(\vec{\pi})$
$\vec{\pi}_1=(\phi,\phi,6)$	+1	–1
$\vec{\pi}_2=(\phi,\phi,5+1)$	+1	–2
$\vec{\pi}_3=(\phi,\phi,4+2)$	+1	–2
$\vec{\pi}_4=(\phi,\phi,4+1+1)$	+1	–3
$\vec{\pi}_5=(\phi,\phi,3+3)$	+1	–2
$\vec{\pi}_6=(\phi,\phi,3+2+1)$	+1	–3
$\vec{\pi}_7=(\phi,\phi,3+1+1+1)$	+1	–4
$\vec{\pi}_8=(\phi,\phi,2+2+2)$	+1	–3
$\vec{\pi}_9=(\phi,\phi,2+2+1+1)$	+1	–4
$\vec{\pi}_{10}=(\phi,\phi,2+1+1+1+1)$	+1	–5
$\vec{\pi}_{11}=(\phi,\phi,1+1+1+1+1+1)$	+1	–6
$\vec{\pi}_{12}=(\phi,6,\phi)$	+1	1
$\vec{\pi}_{13}=(\phi,5+1,\phi)$	+1	2
$\vec{\pi}_{14}=(\phi,4+2,\phi)$	+1	2
$\vec{\pi}_{15}=(\phi,4+1+1,\phi)$	+1	3
$\vec{\pi}_{16}=(\phi,3+3,\phi)$	+1	2
$\vec{\pi}_{17}=(\phi,3+2+1,\phi)$	+1	3
$\vec{\pi}_{18}=(\phi,3+1+1+1,\phi)$	+1	4
$\vec{\pi}_{19}=(\phi,2+2+2,\phi)$	+1	3
$\vec{\pi}_{20}=(\phi,2+2+1+1,\phi)$	+1	4
$\vec{\pi}_{21}=(\phi,2+1+1+1+1,\phi)$	+1	5
$\vec{\pi}_{22}=(\phi,1+1+1+1+1+1,\phi)$	+1	6

INTERNATIONAL JOURNAL of RESEARCH –GRANTHAALAYAH
A knowledge Repository
Science

$\vec{\pi}_{23} = (6,\phi,\phi)$	-1	0
$\vec{\pi}_{24} = (5+1,\phi,\phi)$	$+1$	0
$\vec{\pi}_{25} = (4+2,\phi,\phi)$	$+1$	0
$\vec{\pi}_{26} = (3+2+1,\phi,\phi)$	-1	0
$\vec{\pi}_{27} = (\phi,5,1)$	$+1$	0
$\vec{\pi}_{28} = (\phi,1,5)$	$+1$	0
$\vec{\pi}_{29} = (\phi,4,2)$	$+1$	0
$\vec{\pi}_{30} = (\phi,2,4)$	$+1$	0
$\vec{\pi}_{31} = (\phi,4,1)$	$+1$	1
$\vec{\pi}_{32} = (\phi,4,1+1)$	$+1$	-1
$\vec{\pi}_{33} = (\phi,1,4+1)$	$+1$	-1
$\vec{\pi}_{34} = (\phi,1+1,4)$	$+1$	1
$\vec{\pi}_{35} = (\phi,3,3)$	$+1$	0
$\vec{\pi}_{36} = (\phi,3+2,1)$	$+1$	1
$\vec{\pi}_{37} = (\phi,1,3+2)$	$+1$	-1
$\vec{\pi}_{38} = (\phi,3,2+1)$	$+1$	-1
$\vec{\pi}_{39} = (\phi,2+1,3)$	$+1$	1
$\vec{\pi}_{40} = (\phi,1+3,2)$	$+1$	1
$\vec{\pi}_{41} = (\phi,2,1+3)$	$+1$	-1
$\vec{\pi}_{42} = (\phi,3,1+1+1)$	$+1$	-2
$\vec{\pi}_{43} = (\phi,3+1,1+1)$	$+1$	0
$\vec{\pi}_{44} = (5,\phi,1)$	-1	-1
$\vec{\pi}_{45} = (5,1,\phi)$	-1	1
$\vec{\pi}_{46} = (4,\phi,2)$	-1	-1
$\vec{\pi}_{47} = (4,2,\phi)$	-1	1
$\vec{\pi}_{48} = (\phi,1+1+1,3)$	$+1$	2
$\vec{\pi}_{49} = (\phi,1+1,3+1)$	$+1$	0
$\vec{\pi}_{50} = (\phi,1,3+1+1)$	$+1$	-2
$\vec{\pi}_{51} = (\phi,3+1+1,1)$	$+1$	2

INTERNATIONAL JOURNAL of RESEARCH –GRANTHAALAYAH
A knowledge Repository
Science

$\vec{\pi}_{52} = (\phi,2+2,2)$	+1	1
$\vec{\pi}_{53} = (\phi,2,2+2)$	+1	–1
$\vec{\pi}_{54} = (\phi,2,1+1+1+1)$	+1	–3
$\vec{\pi}_{55} = (\phi,1+1+1+1,2)$	+1	3
$\vec{\pi}_{56} = (\phi,2+1,1+1+1)$	+1	–1
$\vec{\pi}_{57} = (\phi,1+1+1,2+1)$	+1	1
$\vec{\pi}_{58} = (\phi,2+1+1,1+1)$	+1	1
$\vec{\pi}_{59} = (\phi,1+1,2+1+1)$	+1	–1
$\vec{\pi}_{60} = (\phi,1+1+1+1,1+1)$	+1	2
$\vec{\pi}_{61} = (\phi,1+1,1+1+1+1)$	+1	–2
$\vec{\pi}_{62} = (\phi,1+1+1,1+1+1)$	+1	0
$\vec{\pi}_{63} = (\phi,1,1+1+1+1+1)$	+1	–4
$\vec{\pi}_{64} = (\phi,1+1+1+1+1,1)$	+1	4
$\vec{\pi}_{65} = (3,2,1)$	–1	0
$\vec{\pi}_{66} = (3,1,2)$	–1	0
$\vec{\pi}_{67} = (2,3,1)$	–1	0
$\vec{\pi}_{68} = (2,1,3)$	–1	0
$\vec{\pi}_{69} = (1,2,3)$	–1	0
$\vec{\pi}_{70} = (1,3,2)$	–1	0
$\vec{\pi}_{71} = (3,1,1+1)$	–1	–1
$\vec{\pi}_{72} = (3,1+1,1)$	–1	1
$\vec{\pi}_{73} = (2,2+1,1)$	–1	1
$\vec{\pi}_{74} = (2,1,1+2)$	–1	–1
$\vec{\pi}_{75} = (1,1+1+1,1+1)$	–1	1
$\vec{\pi}_{76} = (1,1+1,1+1+1)$	–1	–1
$\vec{\pi}_{77} = (1,1,1+1+1+1)$	–1	–3
$\vec{\pi}_{78} = (1,1+1+1+1,1)$	–1	3
$\vec{\pi}_{79} = (2,1+1,1+1)$	–1	0

INTERNATIONAL JOURNAL of RESEARCH –GRANTHAALAYAH
A knowledge Repository
Science

$\vec{\pi}_{80} = (4,1,1)$	–1	0
$\vec{\pi}_{81} = (3,\phi,3)$	–1	–1
$\vec{\pi}_{82} = (3,3,\phi)$	–1	1
$\vec{\pi}_{83} = (3,1+1+1,\phi)$	–1	3
$\vec{\pi}_{84} = (3,\phi,1+1+1)$	–1	–3
$\vec{\pi}_{85} = (2,2+2,\phi)$	–1	2
$\vec{\pi}_{86} = (2,\phi,2+2)$	–1	–2
$\vec{\pi}_{87} = (2,2+1+1,\phi)$	–1	3
$\vec{\pi}_{88} = (2,\phi,2+1+1)$	–1	–3
$\vec{\pi}_{89} = (2,\phi,2+2)$	–1	–2
$\vec{\pi}_{90} = (2,\phi,1+1+1+1)$	–1	–4
$\vec{\pi}_{91} = (1,1+1+1+1+1,\phi)$	–1	–4
$\vec{\pi}_{92} = (1,\phi,1+1+1+1+1)$	–1	–5
$\vec{\pi}_{93} = (1+2,3,\phi)$	+1	1
$\vec{\pi}_{94} = (1+2,\phi,3)$	+1	–1
$\vec{\pi}_{95} = (3+1,2,\phi)$	+1	1
$\vec{\pi}_{96} = (3+1,\phi,2)$	+1	–1
$\vec{\pi}_{97} = (3+1,1,1)$	+1	0
$\vec{\pi}_{98} = (4+1,1,\phi)$	+1	1
$\vec{\pi}_{99} = (4+1,\phi,1)$	+1	–1
$\vec{\pi}_{100} = (4,1+1,\phi)$	–1	2
$\vec{\pi}_{101} = (4,\phi,1+1)$	–1	–2
$\vec{\pi}_{102} = (3+1,1+1,\phi)$	+1	2
$\vec{\pi}_{103} = (3+1,\phi,1+1)$	+1	–2
$\vec{\pi}_{104} = (2+1,1+1+1,\phi)$	+1	3
$\vec{\pi}_{105} = (2+1,\phi,1+1+1)$	+1	–3
$\vec{\pi}_{106} = (2+1,1,2)$	+1	0
$\vec{\pi}_{107} = (2+1,2,1)$	+1	0

Science
INTERNATIONAL JOURNAL of RESEARCH –GRANTHAALAYAH
A knowledge Repository

$\vec{\pi}_{108} = (1,2+1,2)$	–1	1
$\vec{\pi}_{109} = (1,2,2+1)$	–1	–1
$\vec{\pi}_{110} = (1,2+3,\phi)$	–1	2
$\vec{\pi}_{111} = (1,\phi,2+3)$	–1	–2
$\vec{\pi}_{112} = (\phi,4,2)$	+1	1
$\vec{\pi}_{113} = (2+3,\phi,1)$	+1	–1
$\vec{\pi}_{114} = (2,1+3,\phi)$	–1	2
$\vec{\pi}_{115} = (2,\phi,3+1)$	–1	–2
$\vec{\pi}_{116} = (1,2+2+1,\phi)$	–1	3
$\vec{\pi}_{117} = (1,\phi,2+2+1)$	–1	–3
$\vec{\pi}_{118} = (2+1,1+1,1)$	+1	1
$\vec{\pi}_{119} = (2+1,1,1+1)$	+1	–1
$\vec{\pi}_{120} = (1,1+1,2+1)$	–1	0
$\vec{\pi}_{121} = (1,2+1,1+1)$	–1	0

From this table we have;

$$M_V(0,11,6) = \omega(\vec{\pi}_{23}) + \omega(\vec{\pi}_{24}) + \omega(\vec{\pi}_{25}) + \omega(\vec{\pi}_{26}) + \omega(\vec{\pi}_{27}) + \omega(\vec{\pi}_{28}) + \omega(\vec{\pi}_{29}) + \omega(\vec{\pi}_{30}) + \omega(\vec{\pi}_{35}) +$$
$$\omega(\vec{\pi}_{43}) + \omega(\vec{\pi}_{49}) + \omega(\vec{\pi}_{62}) + \omega(\vec{\pi}_{65}) + \omega(\vec{\pi}_{66}) + \omega(\vec{\pi}_{67}) + \omega(\vec{\pi}_{68}) + \omega(\vec{\pi}_{69}) +$$
$$\omega(\vec{\pi}_{70}) + \omega(\vec{\pi}_{79}) + \omega(\vec{\pi}_{25}) + \omega(\vec{\pi}_{80}) + \omega(\vec{\pi}_{97}) + \omega(\vec{\pi}_{106}) + \omega(\vec{\pi}_{107}) + \omega(\vec{\pi}_{120}) + \omega(\vec{\pi}_{121})$$

= –1+1+1–1+1+1+1+1+1+1+1+1–1–1–1–1–1–1–1–1+1+1+1–1–1 = 1.

$M_V(0,11,6) = 1 = \frac{P(6)}{11}$, where k = 0 and n = 0. Hence the Result.

7. CONCLUSION

In this study we have found the cranks of partitions of integers 7 and 8 respectively and have shown the generating function for M (m, n) and M (o, n). We have established the Dyson's conjectures related to the partition congruences modulo 5, 7 and 11. Finally we have established the Mathematical results of Dyson's conjectures. These results are combinatorial results of Ramanujan's partitions congruences.

8. REFERENCES

[1] G.E. Andrews, The Theory of Partitions, Encyclopedia of Mathematics and its Application, Vol. 2 (G.- C. Rota, ed.), Addison-Wesley, Reading, mass, 1976 (Reissued, Cambridge University Press, London and New York,1985).

[2] G.E. Andrews and F.G. Garvan, Dyson's Crank of a Partition, Bulletin (New series) of the

INTERNATIONAL JOURNAL of
RESEARCH –GRANTHAALAYAH
A knowledge Repository
Science

American Mathematical Society, 18(2): 167–171. (1988).
[3] *A.O.L .Atkin and H.P.F. Swinnerton-Dyer, (1954), Some Properties of Partitions, Proc. London Math . Soc. 3(4): 84–106.*
[4] *F. Dyson, Some guesses in the theory of partitions, (1944), Eureka ,Cambridge, U.K. 8, 10-15.*
[5] *F.G. Garvan, Ramanujan Revisited, Proceeding of the Centenary Conference, University of Illinois, Urban-Champion.(1988).*
[6] *F.G. Garvan, Dyson's Rank Function and Andrews' spt-function, University of Florida, Seminar Paper Presented in the University of Newcastle on 20 August (2013).*
[7] *F.G. Garvan, Generalizations of Dyson's rank, Ph.D. Thesis, Pennsylvania State University, (1986) .*
[8]. *G.H. Hardy, E.M. Wright, (1965). Introduction to the Theory of Numbers, 4th Edition, Oxford, Clarendon Press .*
[9] *K. Mahlburg , Partitions congruences, (2005), Proc. Natl. Acad.Sci.USA102,15373- 15376.*