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PARTITION CONGRUENCES AND DYSON'S RANK

Sabuj Das*[1]
*[1]Senior Lecturer, Department of Mathematics Raozan University College, Bangladesh
*Correspondence Author: sabujdas.ctg@gmail.com

Abstract:
In this article the rank of a partition of an integer is a certain integer associated with the partition. The term has first introduced by freeman Dyson in a paper published in Eureka in 1944. In 1944, F.S. Dyson discussed his conjectures related to the partitions empirically some Ramanujan's famous partition congruences. In 1921, S. Ramanujan proved his famous partition congruences: The number of partitions of numbers 5n+4, 7n+5 and 11n +6 are divisible by 5, 7 and 11 respectively in another way. In 1944, Dyson defined the relations related to the rank of partitions. These are later proved by Atkin and Swinnerton-Dyer in 1954.
The proofs are analytic relying heavily on the properties of modular functions. This paper shows how to generate the generating functions for $N(1,n), N(-1,n), N(2,n), N(m,n), N(0,5,n), N(1,5,n)...$ ***In this paper, we show how to prove the Dyson's conjectures with rank of partitions.***

Keywords:
Modulo, Dyson's conjectures, Rank of partition, Ramanujan's lost Note book, theta series.

1. INTRODUCTION

In this paper we give some related definitions of P(n), rank of partition, $N(1,n), N(-1,n), N(2,n), N(m,n), N(0,5,n), N(m,t,n),...$ z, $(x)_\infty$, $(zx)_\infty$, $(x^n)_m$, $(x^k;x^5)_m$. We discuss the generating functions for N(m, n), N(m, t, n) and associative terms for various integers of m and n and discuss the Dyson's conjectures related to the rank of partitions. We show the Dyson's conjectures by the equating the coefficients of various powers of x of two generating functions. In this paper, we prove some Dyson's conjectures given by

$N(k,5,5n+4)=\dfrac{P(5n+4)}{5};0\le k\le 4,$

$N(k,7,7n+5)=\dfrac{P(7n+5)}{7};0\le k\le 6,\ N(k,11,11n+6)=\dfrac{P(11n+5)}{11};0\le k\le 10,$

$N(m,t,n)=N(t-m,t,n)$, $N(1,5,5n+1)=N(2,5,5n+1)$, $N(1,5,5n+1)=N(2,5,5n+1)$,

$N(0,7,7n+4)=N(1,7,7n+4)$, $N(2,7,7n+1)=N(3,7,7n+1)$.

2. SOME RELATED DEFINITIONS

P(n) [Agarwal,etel (1979)] : The number of partitions of n like 4, 3+1, 2+2,

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2+1+1,1+1+1+1. $\therefore$ P (4) =5.

Rank of partition: The largest part of a partition π minus the number of parts of the partition π (a partition of n).

$N(1,n)$: The number of partitions of n with rank 1.

$N(-1,n)$: The number of partitions of n with rank -1.

$N(2,n)$: The number of partitions of n with rank 2.

$N(m,n)$: The symbol is denoted by the number of partitions of n with rank m.

$N(0,5,n)$: The number of partitions of n with rank congruent to 0 modulo5.

$N(m,t,n)$: The symbol is denoted by the number of partitions of n with rank congruent to m modulo t.

$N(4,5,n)$: The number of partitions of n with rank congruent to 4 modulo5.

$N(2,5,n)$: The number of partitions of n with rank congruent to 2 modulo5.

$N(0,7,n)$: The number of partitions of n with rank congruent to 0 modulo7.

$N(1,7,n)$: The number of partitions of n with rank congruent to 1 modulo7.

Z: The set of complex numbers.

Product notations [Andrews, etel (1989)]:

$$(x)_\infty = (1-x)(1-x^2)(1-x^3)\ldots\infty.$$

$$(zx)_\infty = (1-zx)(1-zx^2)(1-zx^3)\ldots\infty.$$

$$(x^n)_m = (1-x^n)(1-x^{n+1})(1-x^{n+2})\ldots(1-x^{n+m-1}).$$

$$(x^k;x^5)_m = (1-x^k)(1-x^{k+5})(1-x^{k+10})\ldots(1-x^{k+(m-1)5}).$$

Rank of a Partition and Dyson's Conjectures [Garvan (1986)]

The rank of a partition is defined as the largest part of a partition π minus the number of parts of the partition π .Thus the partition 5 + 4+ 1 + 1 of 11 has rank 5–4= 1 and the conjugate partition 4+ 2 + 2 + 2 + 1 has rank 4–5 = -1. i.e. the rank of a partition and that of the conjugate partition differ only in sign. The rank of a partition of 11 belongs to any one of the residues (mod 11) and we have exactly 11 residues.

Remark 1: $N(k,5,5n+4) = \frac{P(5n+4)}{5}; 0 \le k \le 4.$

Proof: We get the list of all partitions of 9 is;

9, 8 + 1, 7 + 2, 7 + 1 + 1, 6 + 3, 6 + 2 + 1, 6 + 1 + 1 + 1, 5 + 4, 5 + 3 + 1,
5 + 2 + 1 + 1, 5 + 2 + 2, 5 + 1 + 1 + 1 + 1,
... ...
2 + 1 + 1 + 1 + 1 + 1 + 1 + 1,1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1.

Hence, there are 30 partitions i.e., P(9) = 30.

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Now there corresponding ranks are;

8, 6, 5, 4, 4, 3, 2, 3, 2, 1, 2, 0, 1, 0

... ...

–1, –2, –3, –4, –5, –6, –8 .

$\therefore$ N(0, 5, 9) = 6, N(1, 5, 9) = 6, N(2, 5, 9) = 6, N(3, 5, 9) = 6, N(4, 5, 9) = 6,

and $\frac{P(9)}{5} = \frac{30}{5} = 6$.

$\therefore$ N(0, 5, 9) = N(1, 5, 9) = N(2, 5, 9) = N(3, 5, 9) = N(4, 5, 9) = 6 = $\frac{P(9)}{5}$.

So, we can say that; N(k, 5, 9) = $\frac{P(9)}{5}$; $0 \le k \le 4$.

Generally we can conclude that; $N(k,5,5n+4) = \frac{P(5n+4)}{5}; 0 \le k \le 4$.

Hence the Remark .

Remark 2: $N(k,7,7n+5) = \frac{P(7n+5)}{7}; 0 \le k \le 6$.

Proof: We get the list of all partitions of 12 is;

12, 11 + 1, 10 + 2, 10 + 1 + 1, 9 + 3, 9 + 2 + 1, 9 + 1 + 1 + 1,

8 + 4, 8 + 3 + 1, 8 + 2 + 1 + 1, 8 + 2 + 2, 7 + 5, 7 + 4 + 1, 7 + 3 + 1 + 1,

... ...

2 + 2 + 3 + 1 + 1 + 1 + 1 + 1, 2 + 2 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1,

2 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1,1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1.

$\therefore$ There are 77 partitions, i.e., P(12) = 77.

Now there corresponding ranks are;

11,	9,	8,	7,	7,	6,	5,	6,	5
4,	5,	3,	5,	4,	3,	2,	1,	4
...	...	...	...	...	...	...	...	...
–4,	–5,	–6,	–7,	–8,	–9,	–11.		

$\therefore$ N(0, 7, 12) = 11, N(1, 7, 12) = 11, N(2, 7, 12) = 11, ... , N(6, 7, 12) = 11.

and $\frac{P(12)}{7} = \frac{77}{7} = 11$.

$\therefore$ N(0, 7, 12) = N(1, 7, 12) = ... = N (6, 7, 12) = 11 = $\frac{P(12)}{7}$.

So we can say that; N(k, 7, 12) = $\frac{P(12)}{7}$; $0 \le k \le 6$.

Generally, we can say that, $N(k,7,7n+5) = \frac{P(7n+5)}{7}; 0 \le k \le 6$. Hence the Remark .

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Remark 3: $N(k,11,11n+6)=\dfrac{P(11n+5)}{11};0\le k\le 10.$

Proof: We get the list of all partitions of 17 is;

17,16+1,15+2,15+1+1,14+3,14+2+1,14+1+1+1,13+4,13+3+1,13+2+2,13+2+1+1,13+1+1+1+1,12+5,12+4+1, …. ,1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1.

Hence, there are 297 partitions i.e., P(17) = 297.

Now there corresponding ranks are;

16,14,13,12,12,11,10,11,10,….,-13,-14,-16.

$$N(0,11,17)=N(1,11,17)=N(2,11,17)=N(3,11,17)=N(4,11,17)=\ldots$$

$$=N(9,11,17)=N(10,11,17)=27=\frac{P(17)}{11}.$$

So, we can say that; $N(k,11,17)=\dfrac{P(17)}{11};0\le k\le 10.$

Generally we can conclude that; $N(k,11,11n+6)=\dfrac{P(11n+6)}{11};0\le k\le 10.$

Hence the Remark.

We discuss some generating functions below [Andrews (1979)]:

The generating function for $N(1,n)$ is given below;

We get, $N(1,n)$ is the number of partitions of *n* with rank 1, like:

n:	type of partitions	$N(1,n)$
1:	none	0
2:	2	1
3:	none	0
4:	3+1	1
5:	3+2	1
6:	4+1+1,3+3	2
…	…	…

We make the expression;

$$N(1,0)+N(1,1)x+N(1,2)x^2+N(1,3)x^3+N(1,4)x^4+\ldots,\text{ where } N(1,0)=0$$

$$=0+x^2+0+x^4+x^5+2x^6+\ldots$$

$$=(x^2-x^3-x^7+x^9+\ldots)\,(1+x+2x^2+3x^3+5x^4+\ldots)$$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}x^{\frac{n}{2}(3n-1)+n}(1-x^n)\prod_{j=1}^{\infty}\frac{1}{1-x^j}\qquad\text{[Garvan (1988)]}$$

$$\therefore\sum_{n=1}^{\infty}(-1)^{n-1}x^{\frac{n}{2}(3n-1)+n}(1-x^n)\prod_{j=1}^{\infty}\frac{1}{1-x^j}=\sum_{n=0}^{\infty}N(1,n)x^n.\qquad(1)$$

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Again, the generating function for $N(-1,n)$ is given below;

We get, $N(-1,n)$ is the number of partitions of *n* with rank -1, like:

n:	type of partitions	$N(-1,n)$
1:	none	0
2:	1+1	1
3:	none	0
4:	2+1+1	1
5:	2+2+1	1
6:	3+1+1+1,2+2+2	2
…	…	…

We make the expression;

$$N(-1,0)+N(-1,1)x+N(-1,2)x^2+N(-1,3)x^3+N(-1,4)x^4+\dots,\text{ where } N(-1,0)=0$$

$$=0+x^2+0+x^4+x^5+2x^6+\dots$$

$$=(x^2-x^3-x^7+x^9+\dots)\ (1+x+2x^2+3x^3+5x^4+\dots)$$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}x^{\frac{n}{2}(3n-1)+n}(1-x^n)\prod_{j=1}^{\infty}\frac{1}{1-x^j}\quad \text{[Garvan (2013)]}$$

$$\therefore \sum_{n=1}^{\infty}(-1)^{n-1}x^{\frac{n}{2}(3n-1)+n}(1-x^n)\prod_{j=1}^{\infty}\frac{1}{1-x^j}=\sum_{n=0}^{\infty}N(-1,n)x^n. \qquad (2)$$

From (1) and (2) we get, $N(1,n)=N(-1,n)$, [by equating the coefficient of x^n].

The generating function for $N(2,n)$ is given below;

We get, $N(2,n)$ is the number of partitions of *n* with rank 2, like:

n:	type of partitions	$N(2,n)$
1:	none	0
2:	none	0
3:	3	1
4:	none	0
5:	4+1	1
…	…	…

We make the expression;

$$N(2,0)+N(2,1)x+N(2,2)x^2+N(2,3)x^3+N(2,4)x^4+\dots,\text{ where } N(2,0)=0$$

$$=0+x^3+x^5+x^6+\dots$$

$$=(x^3-x^4-x^9+\dots)\ (1+x+2x^2+3x^3+5x^4+\dots)$$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}x^{\frac{n}{2}(3n-1)+2n}(1-x^n)\prod_{j=1}^{\infty}\frac{1}{1-x^j}=\sum_{n=0}^{\infty}N(2,n)x^n.$$

$$\therefore \sum_{n=1}^{\infty}(-1)^{n-1}x^{\frac{n}{2}(3n-1)+2n}(1-x^n)\prod_{j=1}^{\infty}\frac{1}{1-x^j}=\sum_{n=0}^{\infty}N(2,n)x^n. \qquad (3)$$

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From (1),(2) and (3) we can conclude that,

$$\sum_{n=1}^{\infty}(-1)^{n-1}x^{\frac{n}{2}(3n-1)+|m|n}(1-x^n)\prod_{j=1}^{\infty}\frac{1}{1-x^j}=\sum_{n=0}^{\infty}N(m,n)x^n.$$

The generating function for $N(0,5,n)$ is given below:

We get, $N(0,5,n)$ is the number of partitions of *n* with rank congruent to 0 modulo5, like:

n:	type of partitions	$N(0,5,n)$
1:	1	1
2:	none	0
3:	2+1	1
4:	2+2	1
…	…	…

We make the expression;

$N(0,5,0)+N(0,5,1)x+N(0,5,2)x^2+N(0,5,3)x^3+\ldots$, where $N(0,5,0)=0$

$$=0+x+x^3+x^4+\ldots$$

$$=\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}(-1)^n x^{\frac{n}{2}(3n+1)}(1+x^{5n})(1-x^{5n})^{-1}(1+x+2x^2+3x^3+5x^4+\ldots)$$

$$\therefore \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}(-1)^n x^{\frac{n}{2}(3n+1)}(1+x^{5n})(1-x^{5n})^{-1}\prod_{j=1}^{\infty}(1-x^j)^{-1}=\sum_{n=0}^{\infty}N(0,5,n)x^n.$$

Remark 4: $N(m,t,n)=N(t-m,t,n)$

Proof: The generating function for $N(1,5,n)$ is given below:

We get, $N(1,5,n)$ is the number of partitions of *n* with rank congruent to 1 modulo5, like:

n:	type of partitions	$N(1,5,n)$
1:	none	0
2:	2	1
3:	none	0
4:	3+1	1
…	…	…

We make the expression;

$N(1,5,0)+N(1,5,1)x+N(1,5,2)x^2+N(1,5,3)x^3+\ldots$, where $N(1,5,0)=0$

$$=0+0+x^2+0+x^4+\ldots$$

$$=\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}(-1)^n x^{\frac{n}{2}(3n+1)}(x^n+x^{4n})(1-x^{5n})^{-1}\prod_{j=1}^{\infty}\frac{1}{1-x^j}$$

$$\therefore \sum_{n\neq-\infty}^{\infty}(-1)^n x^{\frac{n}{2}(3n+1)}(x^n+x^{4n})(1-x^{5n})^{-1}\prod_{j=1}^{\infty}(1-x^j)^{-1}=\sum_{n=0}^{\infty}N(1,5,n)x^n. \quad (4)$$

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Again the generating function for $N(4,5,n)$ is given below:

We get, $N(4,5,n)$ is the number of partitions of *n* with rank congruent to 1 modulo5, like:

n:	type of partitions	$N(1,5,n)$
1:	none	0
2:	2	1
3:	none	0
4:	3+1	1
…	…	…

We make the expression;

$N(4,5,0)+N(4,5,1)x+N(4,5,2)x^2+N(4,5,3)x^3+\ldots$, where $N(4,5,0)=0$

$$=0+0+x^2+0+x^4+\ldots$$

$$=\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}(-1)^n x^{\frac{n}{2}(3n+1)}\left(x^n+x^{4n}\right)\left(1-x^{5n}\right)^{-1}\prod_{j=1}^{\infty}\frac{1}{1-x^j}$$

$$\therefore \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}(-1)^n x^{\frac{n}{2}(3n+1)}\left(x^n+x^{4n}\right)\left(1-x^{5n}\right)^{-1}\prod_{j=1}^{\infty}\left(1-x^j\right)^{-1}=\sum_{n=0}^{\infty}N(4,5,n)x^n. \quad (5)$$

From (4) and (5) we get;

$N(1,5,n)=N(4,5,n)$

$\therefore N(1,5,n)=N(5-1,5,n)$.

It follows that, $N(m,5,n)=N(5-m,5,n)$; $m=[0,4]$.

Generally, we can conclude that, $N(m,t,n)=N(t-m,t,n)$. Hence the Remark .

Remark 5: $N(1,5,5n+1)=N(2,5,5n+1)$

Proof: The generating function for $N(2,5,n)$ is given below:

We get, $N(2,5,n)$ is the number of partitions of *n* with rank congruent to 2 modulo5, like:

n:	types of partitions	$N(2,5,n)$
1:	none	0
2:	none	0
3:	3	1
4:	1+1+1+1	1
…	…	…

We make the expression;

$N(2,5,0)+N(2,5,1)x+N(2,5,2)x^2+N(2,5,3)x^3+\ldots$, where $N(2,5,0)=0$

$$=0+0+x^2+x^3+\ldots$$

$$=\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}(-1)^n x^{\frac{n}{2}(3n+1)}\left(x^{2n}+x^{3n}\right)\left(1-x^{5n}\right)^{-1}\prod_{j=1}^{\infty}\frac{1}{1-x^j}$$

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$$\therefore \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (-1)^n x^{\frac{n}{2}(3n+1)} \left(x^{2n} + x^{3n}\right)\left(1-x^{5n}\right)^{-1} \prod_{j=1}^{\infty} \frac{1}{1-x^j} = \sum_{n=0}^{\infty} N(2,5,n)x^n$$

$$= x^3 + x^4 + 2x^6 + 2x^7 + 5x^8 + 6x^9 + 8x^{10} + 11x^{11} + \ldots$$

$$= x^3 + x^4 + N(2,5,6)x^6 + 2x^7 + 5x^8 + 6x^9 + 8x^{10} + N(2,5,11)x^{11} + \ldots \quad (6)$$

Again from (4) we get;

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (-1)^n x^{\frac{n}{2}(3n+1)} \left(x^{n} + x^{4n}\right)\left(1-x^{5n}\right)^{-1} \prod_{j=1}^{\infty} \left(1-x^j\right)^{-1} = \sum_{n=0}^{\infty} N(1,5,n)x^n .$$

$$= x^2 + 0.x^3 + x^4 + 2x^5 + 2x^6 + \ldots + 9x^{10} + 11x^{11} + \ldots$$

$$= x^2 + x^4 + 2x^5 + N(1,5,6)x^6 + \ldots + 9x^{10} + N(1,5,11)x^{11} + \ldots \quad (7)$$

From (6) and (7) we get;

$N(1,5,6) = N(2,5,6)$, $N(1,5,11) = N(2,5,11)$, $N(1,5,16) = N(2,5,16)$, …

Generally, we can conclude that; $N(1,5,5n+1) = N(2,5,5n+1)$. Hence the Remark .

Remark 6: $N(1,5,5n+1) = N(2,5,5n+1)$

Proof: The generating function for $N(2,5,n)$ is given below:

We get, $N(2,5,n)$ is the number of partitions of *n* with rank congruent to 2 modulo5, like:

n:	types of partitions	$N(2,5,n)$
1:	none	0
2:	none	0
3:	3	1
4:	1+1+1+1	1
…	…	…

We make the expression;

$N(2,5,0) + N(2,5,1)x + N(2,5,2)x^2 + N(2,5,3)x^3 + \ldots$, where $N(2,5,0) = 0$

$$= 0 + 0 + x^2 + x^3 + \ldots$$

$$= \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (-1)^n x^{\frac{n}{2}(3n+1)} \left(x^{2n} + x^{3n}\right)\left(1-x^{5n}\right)^{-1} \prod_{j=1}^{\infty} \frac{1}{1-x^j}$$

$$\therefore \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (-1)^n x^{\frac{n}{2}(3n+1)} \left(x^{2n} + x^{3n}\right)\left(1-x^{5n}\right)^{-1} \prod_{j=1}^{\infty} \frac{1}{1-x^j} = \sum_{n=0}^{\infty} N(2,5,n)x^n$$

$$= x^3 + x^4 + 2x^6 + 2x^7 + 5x^8 + 6x^9 + 8x^{10} + 11x^{11} + \ldots$$

$$= x^3 + x^4 + N(2,5,6)x^6 + 2x^7 + 5x^8 + 6x^9 + 8x^{10} + N(2,5,11)x^{11} + \ldots \quad (8)$$

Again from (4) we get;

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$$\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}(-1)^n x^{\frac{n}{2}(3n+1)}\left(x^n+x^{4n}\right)\left(1-x^{5n}\right)^{-1}\prod_{j=1}^{\infty}\left(1-x^j\right)^{-1}=\sum_{n=0}^{\infty}N(1,5,n)x^n.$$

$$=x^2+0.x^3+x^4+2x^5+2x^6+\ldots+9x^{10}+11x^{11}+\ldots$$

$$=x^2+x^4+2x^5+N(1,5,6)x^6+\ldots+9x^{10}+N(1,5,11)x^{11}+\ldots \quad (9)$$

From (8) and (9) we get;

$N(1,5,6)=N(2,5,6)$, $N(1,5,11)=N(2,5,11)$, $N(1,5,16)=N(2,5,16)$, …

Generally, we can conclude that; $N(1,5,5n+1)=N(2,5,5n+1)$. Hence the Remark .

Remark 7: $N(0,7,7n+4)=N(1,7,7n+4)$

Proof: The generating function for $N(0,7,n)$ is given below:

We get, $N(0,7,n)$ is the number of partitions of *n* with rank congruent to 0 modulo7, like:

n:	types of partitions	$N(0,7,n)$
1:	1	1
2:	none	0
3:	2+1	1
4:	2+2	1
…	…	…

We make the expression;

$N(0,7,0)+N(0,7,1)x+N(0,7,2)x^2+N(0,7,3)x^3+N(0,7,4)x^4+\ldots+N(0,7,11)x^{11}+\ldots$,

where $N(0,7,0)=0$

$$=x+x^3+x^4+x^5+x^6+3x^7+\ldots+8x^{11}+\ldots$$

$$=\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}(-1)^n x^{\frac{n}{2}(3n+1)}\left(1+x^{7n}\right)\left(1-x^{7n}\right)^{-1}\prod_{j=1}^{\infty}\frac{1}{1-x^j}$$

$$\therefore \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}(-1)^n x^{\frac{n}{2}(3n+1)}\left(1+x^{7n}\right)\left(1-x^{7n}\right)^{-1}\prod_{j=1}^{\infty}\frac{1}{1-x^j}=\sum_{n=0}^{\infty}N(0,7,n)x^n. \quad (10)$$

Again, the generating function for $N(1,7,n)$ is given below:

We get, $N(1,7,n)$ is the number of partitions of *n* with rank congruent to 1 modulo7, like:

n:	types of partitions	$N(1,7,n)$
1:	none	0
2:	2	1
3:	none	0
4:	3+1	1
…	…	…

We make the expression;

$N(1,7,0)+N(1,7,1)x+N(1,7,2)x^2+N(1,7,3)x^3+N(1,7,4)x^4+\ldots+N(1,7,11)x^{11}+\ldots$,

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where $N(1,7,0)=0$

$$=0+x^2+x^4+x^5+\ldots+8x^{11}+\ldots$$

$$=\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}(-1)^n x^{\frac{n}{2}(3n+1)}\left(x^n+x^{6n}\right)\left(1-x^{7n}\right)^{-1}[1+x+2x^2+3x^3+5x^4+\ldots]$$

$$\therefore \sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}(-1)^n x^{\frac{n}{2}(3n+1)}\left(x^n+x^{6n}\right)\left(1-x^{7n}\right)^{-1}\prod_{j=1}^{\infty}\frac{1}{1-x^j}=\sum_{n=0}^{\infty}N(1,7,n)x^n . \quad (11)$$

From (10) and (11) we get;

$N(0,7,4)=N(1,7,4)$, $N(0,7,11)=N(1,7,11)$,...

Generally, we can conclude that; $N(0,7,7n+4)=N(1,7,7n+4)$. Hence the Remark.

Remark 8: $N(2,7,7n+1)=N(3,7,7n+1)$

Proof: The generating function for $N(2,7,n)$ is given below:

We get, $N(2,7,n)$ is the number of partitions of *n* with rank congruent to 2 modulo7, like:

n:	types of partitions	$N(2,7,n)$
1:	none	0
2:	none	0
3:	3	1
4:	none	0
5:	4+1	1
…	…	…

We make the expression;

$$N(2,7,0)+N(2,7,1)x+N(2,7,2)x^2+N(2,7,3)x^3+N(2,7,4)x^4+N(2,7,8)x^8+\ldots,$$

where $N(2,7,0)=0$

$$=x^3+x^5+x^6+2x^7+3x^{11}+\ldots$$

$$=\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}(-1)^n x^{\frac{n}{2}(3n+1)}\left(x^{2n}+x^{5n}\right)\left(1-x^{7n}\right)^{-1}[1+x+2x^2+3x^3+5x^4+\ldots]$$

$$\therefore \sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}(-1)^n x^{\frac{n}{2}(3n+1)}\left(x^{2n}+x^{5n}\right)\left(1-x^{7n}\right)^{-1}\prod_{j=1}^{\infty}\frac{1}{1-x^j}=\sum_{n=0}^{\infty}N(2,7,n)x^n . \quad (12)$$

Again, the generating function for $N(3,7,n)$ is given below:

We get, $N(3,7,n)$ is the number of partitions of *n* with rank congruent to 3 modulo7, like:

n:	types of partitions	$N(3,7,n)$
1:	none	0
2:	none	0
3:	none	0

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4: 4 1
… … …

We make the expression;

$$N(3,7,0)+N(3,7,1)x+N(3,7,2)x^2+N(3,7,3)x^3+N(3,7,4)x^4+\ldots+N(3,7,8)x^8+\ldots,$$

where $N(3,7,0)=0$

$$=x^4+x^6+2x^7+3x^8+\ldots$$

$$=\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}(-1)^n x^{\frac{n}{2}(3n+1)}\left(x^{3n}+x^{4n}\right)\left(1-x^{7n}\right)^{-1}[1+x+2x^2+3x^3+5x^4+\ldots]$$

$$\therefore \sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}(-1)^n x^{\frac{n}{2}(3n+1)}\left(x^{3n}+x^{4n}\right)\left(1-x^{7n}\right)^{-1}\prod_{j=1}^{\infty}\frac{1}{1-x^j}=\sum_{n=0}^{\infty}N(3,7,n)x^n. \quad (13)$$

From (12) and (13) we get;

$N(2,7,1)=N(3,7,1),\ N(2,7,8)=N(3,7,8)$,...

Generally, we can conclude that; $N(2,7,7n+1)=N(3,7,7n+1)$. Hence the Remark.

3. CONCLUSION

In this study we have established the Dyson's conjectures related to the Ramanujan's famous partition congruences and have shown the generating functions for $N(1,n), N(-1,n), N(2,n), N(m,n), N(0,5,n), N(1,5,n)\ldots$ In this paper, we have proved the Dyson's conjectures with rank of partitions. Some of the Conjectures are

$$N(k,5,5n+4)=\frac{P(5n+4)}{5};0\le k\le 4,\ N(k,7,7n+5)=\frac{P(7n+5)}{7};0\le k\le 6,$$

$$N(k,11,11n+6)=\frac{P(11n+5)}{11};0\le k\le 10,\ N(m,t,n)=N(t-m,t,n)\ldots$$

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5. REFERENCES

[1] Andrews, G.E. and Garvan, F.G. ,Ramanujan's Lost Notebook VI; the Mock Theta Conjectures, Advances in Math., 73, (1989), 242–255.

[2] Agarwal, A.K. ,Partitions Yesterday and Today, New Zealand Math. Soc., Wellington,(1979).

[3] Andrews, G.E. (1979), An Introduction to Ramanujan's Lost Notebook, Amer. Math. Monthly, 86: 89–108.

[4] A.O.L.Atkin and H.P.F.Swinnerton-Dyer, Some properties of partitions,proc. London

INTERNATIONAL JOURNAL of RESEARCH –GRANTHAALAYAH
A knowledge Repository
Science

Math.soc,(3),4 (1954).84-106.

[5] F.J.Dyson, Some guesses in the theory of partitions, Eureka,8 (1944),10-15.

[6] Garvan, F.G.,Generalizations of Dyson's Rank, Ph. D. thesis, Pennsylvania State University,(1986).

[7] Garvan, F.G. (1988), Ramanujan Revisited, Proceeding of the Centenary Conference, University of Illinois, Urban-Champion .

[8] Garvan, F.G. (2013), Dyson's Rank Function and Andrews' spt-function, University of Florida, Seminar Paper Presented in the University of Newcastle on 20August (2013) .

[9] S.Ramanujan, Congruences properties of partitions, Math. J.9 (1921), 147-153.