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NOVEL WAY OF DETERMINING SUM OF K^TH POWERS OF NATURAL NUMBERS

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ABSTRACT

Since ancient times, mathematicians across the world have worked on different methods to find the sum of powers of natural numbers. In this paper, we are going to present the relationship between sum of kth powers of natural numbers and sum of (k−1)^th powers of natural numbers using the differential operator and associated recurrence relation. Interestingly, the Bernoulli numbers which occur frequently in mathematical analysis, play an important role in establishing this relationship.

Keywords: Sum of K^th Powers of Natural Numbers, Differentiation, Bernoulli Numbers, Faulhaber's Triangle

1. INTRODUCTION

1.1. DEFINITION

Let us denote the sum of k^{th} powers of first n natural numbers by

$$S_k(n) = 1^k + 2^k + \cdots + n^k \tag{1}$$

We notice that, $S_0(n) = n$

1.2. DIFFERENTIATION OF $S_k(n)$ FOR $K \geq 0$

In view of formulas presented in Senthil et al. (2014) , we know that $S_k(n)$ is a polynomial in n of degree $k + 1$. Hence $S_k(n)$ is differentiable for each $k \geq 0$. We now differentiate $S_k(n)$ for few values of k to notice some pattern.

For $k = 0$, we know that $S_0(n) = n$

Hence, $\frac{d}{dn}(S_0(n)) = 1$ (2)

Now, for k = 1, $S_1(n) = 1 + 2 + \cdots + n = \frac{1}{2}n^2 + \frac{1}{2}n$

Differentiating and simplifying we get

$$\frac{d}{dn}(S_1(n)) = n + \frac{1}{2} = S_0(n) + \frac{1}{2} \quad (3)$$

For k = 2, $S_2(n) = 1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3$

Differentiating and simplifying we get

$$\frac{d}{dn}(S_2(n)) = \frac{1}{6} + n + n^2 = 2S_1(n) + \frac{1}{6} \quad (4)$$

For k = 3, $S_3(n) = 1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^2 + \frac{1}{2}n^3 + \frac{1}{4}n^4$

Differentiating and simplifying we get

$$\frac{d}{dn}(S_3(n)) = \frac{1}{2}n + \frac{3}{2}n^2 + n^3 = 3S_2(n) \quad (5)$$

For k = 4, $S_4(n) = 1^4 + 2^4 + \cdots + n^4 = -\frac{1}{30}n + \frac{1}{3}n^3 + \frac{1}{2}n^4 + \frac{1}{5}n^5$

Differentiating and simplifying we get

$$\frac{d}{dn}(S_4(n)) = -\frac{1}{30} + n^2 + 2n^3 + n^4 = 4S_3(n) - \frac{1}{30} \quad (6)$$

By observing equations from (3) to (6), we could see that differential of sum of k^{th} powers of natural numbers is equal to k times sum of $(k-1)^{th}$ powers of natural numbers plus a constant. But what are those constants? To see this, we make the following definition.

1.3. DEFINITION OF BERNOULLI NUMBERS

Bernoulli Numbers are numbers which occur as coefficients of $\frac{x^n}{n!}$ in the Taylor's series expansion of $\frac{x}{e^x - 1}$ about $x = 0$. We denote the nth Bernoulli Number by B_n. For knowing more about Bernoulli numbers and their properties see Sivaraman (2020)

Thus, by definition we get $\frac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$ (7)

We notice that the constant term of $\frac{x}{e^x - 1}$ is 1 and so we obtain $B_0 = 1$.

In view of Sivaraman (2020), we know that the Bernoulli numbers satisfy the equation

$$\sum_{j=0}^{n} \binom{n+1}{j} B_j = 0 \quad (8)$$

Using the fact that $B_0 = 1$ and (8), the first few Bernoulli numbers are given by

$$\left.\begin{matrix} B_0 = 1, B_1 = \pm\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}, B_7 = 0, B_8 = -\frac{1}{30}, B_9 = 0 \\ B_{10} = \frac{5}{66}, B_{11} = 0, B_{12} = -\frac{691}{2730}, B_{13} = 0, B_{14} = \frac{7}{6}, B_{15} = 0, B_{16} = -\frac{3617}{510}, \ldots \end{matrix}\right\} \quad (9)$$

1.4. CONSTRUCTION OF FAULHABER'S TRIANGLE

We now construct a triangle of numbers whose entries are denoted by $T(p,q)$ where $q = 0,1,2,3,\ldots,p$. Here p denote the row beginning from 0 and q denote the column beginning with 0 and ending with p for given value of p. The entry of row 0 should be 1. That is, $T(0,0) = 1$. Assuming that row $p - 1$ is known, the entries in the pth row is given by the formula

$$T(p,q) = T(p-1, q-1) \times \frac{p}{q+1} \quad (10)$$

Equation (10) is used to compute $T(p,1)$ up to $T(p,p)$.

The entries in the p^{th} row, first column is calculated in such a way that the row sum is always 1. That is, we should have

$$T(p,0) = 1 - \sum_{q=1}^{p} T(p,q) \quad (11)$$

Equations (10) and (11) are used to construct the following triangle up to first eleven rows.

Figure 1

	Col 0	Col 1	Col 2	Col 3	Col 4	Col 5	Col 6	Col 7	Col 8	Col 9	Col 10
Row 0	1										
Row 1	$\frac{1}{2}$	$\frac{1}{2}$									
Row 2	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$								
Row 3	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$							
Row 4	$\frac{-1}{30}$	0	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{5}$						
Row 5	0	$\frac{-1}{12}$	0	$\frac{5}{12}$	$\frac{1}{2}$	$\frac{1}{6}$					
Row 6	$\frac{1}{42}$	0	$\frac{-1}{6}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{7}$				
Row 7	0	$\frac{1}{12}$	0	$\frac{-7}{24}$	0	$\frac{7}{12}$	$\frac{1}{2}$	$\frac{1}{8}$			
Row 8	$\frac{-1}{30}$	0	$\frac{2}{9}$	0	$\frac{-7}{15}$	0	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{9}$		
Row 9	0	$\frac{-3}{20}$	0	$\frac{1}{2}$	0	$\frac{-7}{10}$	0	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{10}$	
Row 10	$\frac{5}{66}$	0	$\frac{-1}{2}$	0	1	0	-1	0	$\frac{5}{6}$	$\frac{1}{2}$	$\frac{1}{11}$

Figure 1 Faulhaber Triangle

From (9) and column 0 of Figure 1, we notice that $T(k,0) = B_k$, where B_k is the k^{th} Bernoulli number. For knowing more about Faulhaber's Triangle and its entries see Sivaraman (2020).

Generalizing equations (3) to (6), I now prove the following important theorem.

1.5. THEOREM 1

If $S_k(n)$ is sum of k^{th} powers of natural numbers, then

$$\frac{d}{dn}(S_k(n)) = kS_{k-1}(n) + B_k \tag{12}$$

where B_k is the k^{th} Bernoulli number.

Proof: In view of Faulhaber's Formula presented in Sivaraman (2020), we notice that

$$S_k(n) = \frac{n^{k+1}}{k+1} + \frac{n^k}{2} + \frac{k.n^{k-1}}{12} + c_{k-3}n^{k-3} + c_{k-5}n^{k-5} + ... + c_2 n^2 + c_1 n \tag{13}$$

Differentiating the expression on both sides of (13) and simplifying we get

$$\frac{d}{dn}(S_k(n)) = \frac{1}{k+1}(k+1)n^k + \frac{1}{2}kn^{k-1} + \frac{k}{12}(k-1)n^{k-2} + c_{k-3}(k-3)n^{k-4} + c_{k-5}(k-5)n^{k-6} + \cdots + c_3(3n^2) + c_2(2n) + c_1$$

$$= k\begin{bmatrix} \frac{n^k}{k} + \frac{1}{2}n^{k-1} + \frac{1}{12}(k-1)n^{k-2} + c_{k-3}\frac{(k-3)}{k}n^{k-4} \\ +c_{k-5}\frac{(k-5)}{k}n^{k-6} + \cdots + c_3\frac{3}{k}n^2 + c_2\frac{2}{k}n \end{bmatrix} + c_1 \quad (14)$$

But from (13), we notice that

$$S_{k-1}(n) = 1^{k-1} + 2^{k-1} + \cdots + n^{k-1}$$

$$= \frac{n^k}{k} + \frac{n^{k-1}}{2} + \frac{(k-1).n^{k-2}}{12} + b_{k-4}n^{k-4} + b_{k-6}n^{k-6} + \ldots + b_2n^2 + b_1n \quad (15)$$

We now notice that the coefficients in $S_k(n)$ and $S_{k-1}(n)$ in terms of entries of Faulhaber's Triangle are given by

$$c_m = T(k, m-1)(16), b_m = T(k-1, m-1) \quad (17)$$

Now using (10), (16) and (17), we deduce the following

$$\left.\begin{aligned} & b_{k-4} = T(k-1, k-5) = T(k, k-4) \times \frac{k-3}{k} = c_{k-3} \times \frac{k-3}{k} \\ & b_{k-6} = T(k-1, k-7) = T(k, k-6) \times \frac{k-5}{k} = c_{k-5} \times \frac{k-5}{k} \\ & \ldots\ldots\ldots\ldots\ldots \\ & b_2 = T(k-1,1) = T(k,2) \times \frac{3}{k} = c_3 \times \frac{3}{k} \\ & b_1 = T(k-1,0) = T(k,1) \times \frac{2}{k} = c_2 \times \frac{2}{k} \\ & c_1 = T(k,0) = B_k \end{aligned}\right\} \quad (18)$$

Substituting (15) and (18) in (14), we get

$$\frac{d}{dn}(S_k(n)) = k\left[\frac{n^k}{k} + \frac{n^{k-1}}{2} + \frac{(k-1).n^{k-2}}{12} + b_{k-4}n^{k-4} + b_{k-6}n^{k-6} + \ldots + b_2n^2 + b_1n\right] + B_k$$

Hence, we obtain $\frac{d}{dn}(S_k(n)) = kS_{k-1}(n) + B_k$

This completes the proof.

2. CONCLUSION

In this paper, using the concepts of Bernoulli numbers and Faulhaber's Triangle, we have provided a novel method by proving a theorem that the derivative of sum of k^{th} powers of first n natural numbers is k times the sum of $(k-1)^{th}$ powers of first n natural numbers plus the k^{th} Bernoulli number. This differential recurrence relation between successive powers of sum of first n natural numbers is very important in the sense that it helps us to obtain $S_k(n)$ knowing $S_{k-1}(n)$ and Bernoulli numbers.

CONFLICT OF INTERESTS

None.

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