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# GRAPHS WITH THE BURNING NUMBERS EQUAL THREE

Wen Li [1]

[1] School of Mathematics and Statistics, Qinghai Minzu University, Xining, Qinghai 810008, China

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**Corresponding Author**

Wen Li, liwzjn@163.com

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## ABSTRACT

The concept of burning number is inspired by the firefighting problem, which is a new measure to describe the speed of information spread. For a general non-trivial connected graph $G$, its burning number $b(G) \geq 2$, and $b(G) = 2$ if and only if the maximum degree of $G$ is $|G| - 1$or $|G| - 2$. In this paper, we characterize graphs with burning number $b(G) = 3$.

**Keywords:** Burning Sequence, Maximum Degree, Burning Number

## 1. INTRODUCTION

Recently, a new graph process, defined as graph burning, which is motivated by contagion processes of graphs such as graph searching paradigms (Firefighter Bonato and Nowakowski (2011)) and graph bootstrap percolation Balogh et al. (2012) is proposed by Bonato et al. in Bonato et al. (2014). The purpose of graph burning is to burn all the vertices as quickly as possible. All the graphs considered in this paper are simple, undirected, and finite.

Let $G$ be a graph. Then the graph burning process of $G$ is a discrete time which defined as follows.

Step 1: At time $t = 0$. All the vertices in this time are unburned.

Step 2: At time $t \geq 1$. One can choose a new unburned vertex $v$ (if such a vertex is available) to burning. And the chosen vertex $v$ is called a source of fire.

If a vertex $v$ is burned, then it must keep burning state until the burning process is finished.

Step 3: At time $t+1$. All the unburned neighbors of vertex $v$ are burned.

Step 4: If all the vertices of $G$ are burned, then the process ends; Otherwise, turn to Step 2.

The vertex which burned in the time $t$ is denoted by $x_i$. If a graph $G$ is burned in $k$ times, then a burning sequence of $G$ is construct by the sources of fire in each time, denoted by $(x_1, x_2, \cdots, x_k)$. The burning number $b(G)$ of a graph $G$, is the length of a shortest burning sequence of $G$. Furthermore, the shortest burning sequence of $G$is defined as optimal burning sequence.

It follows from the definition of $b(G)$ that the burning number $b(G)$ of a graph $G$ also is the minimum size of the sources of fire after the whole burning process finished. It is clear that the burning number of the star graph $K_{1,s}$is $b(K_{1,s}) = 2$and the complete graph$K_n$ is $b(K_n) = 2$.

Figure 1 indicates the burning number of $P_5$ is $b(P_5) = 3$and$(v_1, v_3, v_5)$ is one of its optimal burning sequences.

**Figure 1**

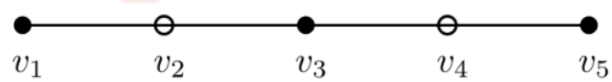


**Figure 1** Burning the path $\boldsymbol{P_5}$ (the black vertices represent the sources of fire)

As proven in Bonato et al. (2015), determining the burning number of a graph $G$ is NP-complete, even for some special graphs such as planar, disconnected, or bipartite graphs. So, it is interesting to study the sharp bounds of the burning number for any connected graphs and characterise the extremal graph. For a general non-trivial connected graph $G$, its burning number $b(G) \geq 2$, and in Bonato and Nowakowski (2011), the authors showed that $b(G) = 2$ if and only if the maximum degree of $G$ is $|G|-1$ or $|G|-2$. In this paper, we consider some sufficient condition on maximum degree of a graph to have $b(G) = 3$. First, we list some useful notations and known results.

Suppose $k$ is a positive integer and $v$ is a vertex of $G = (V(G), E(G))$, then the set $N_k[v] = \{u \in V(G): d(u,v) \leq k\}$ is the $k-th$ closed neighbourhood of a vertex $v$.

**Proposition 1.1.[1]** If a vertex sequence $(x_1, x_2, \cdots, x_k)$ in graph $G$ such that $N_{k-1}[x_1] \cup N_{k-2}[x_2] \cup \cdots \cup N_0[x_k] = V(G)$, then $b(G) \leq k$.

**Proposition 1.2.[1]** For a graph $G$, if $T$ is a spanning subtree of $G$ then $b(G) = min\{\, b(T): T is a spanning tree of G\}$.

**Proposition 1.3. [1]** Let $H$be an isometric subtree of a graph $G$. Then $b(H) \leq b(G)$.

**Proposition 1.4. [1]** If $H$ is a subtree of tree $T$, then $b(H) \leq b(T)$.

**Proposition 1.5. [1]** Let $P_n$ and $C_n$ are path and cycle with order $n$, respectively. Then $b(P_n) = b(C_n) = \lceil \sqrt{n} \rceil$.

**Proposition 1.6. [1]** Let $G$ be a graph with diameter $d$ and radius $r$. Then $\lceil\lceil\sqrt{d+1}\rceil \leq b(G) \leq r+1$.

Let $r$ be a positive integer and $P_i$ denote the path with order $r$. A spider graph $SP(s,r)$ is obtained by connecting a disjoint union of paths $\{P_i\}_{i=1}^{s}$, with $s \geq 3$, to a vertex $u$. The maximum degree of the spider graph $SP(s,r)$ is the degree of vertex $u$ that is equal to $s$. Each subgraph $u \cup P_i$ that has same length $r$ is called an arm of $SP(s,r)$.

**Proposition 1.7. [1]** Let $SP(s,r)$ be a spider graph with maximum degree $s$ and same length $r$ of arms. If $s \geq r$, then $b(SP(s,r)) = r+1$.

**Proposition 1.8. [1]** Let $G$ be a graph with order at least 2. Then $b(G) = 2$ if and only if the maximum degree of $G$ is $|G|-1$or $|G|-2$.

## 2. GRAPHS $G$ WITH $b(G) = 3$

In this section, we characterise graphs with burning number $b(G) = 3$. First, we consider a sufficient condition on the maximum degree of graphs with burning number $b(G) = 3$.

**Theorem 2.1.** Let $T$ be a tree with order $n$ and maximum degree $\Delta$. If $n-6 \leq \Delta \leq n-3$, then $b(T) = 3$.

Proof. Let $T$ be a tree with order $n$ and maximum degree $\Delta$. We first consider the case for $\Delta = n-6$. Suppose the vertex $v$ in $T$ with $d(v) = n-6$, then there exist five vertices are not adjacent to $v$ in $T$, named $v_1, v_2, v_3, v_4\ v_5$, respectively, and let $S = \{v_1, v_2, v_3, v_4, v_5\}$. Since $T$ is connected, then we have $S \cap N_2(v) \neq$ ⍰. Now we distinguish five cases to complete the proof:

**Case 1.** $|S \cap N_2(v)| = 5$.

This means that $S \cap N_2(v) = S$ and $V(T) = N_2(v)$. Let $x_1 = v$ and $x_2 = v_i$, $x_3 = v_j$ for $1 \leq i \neq j \leq 5$. It is clear that $N_2[x_1] \cup N_1[x_2] \cup N_0[x_3] = V(T)$.

**Case 2.** $|S \cap N_2(v)| = 4$.

Without loss generality, suppose $v_5 \notin S \cap N_2(v)$, this means that $d(v, v_5) = 3$. Let $x_1 = v$, $x_3 = v_5$and $x_2 = v_i$ for $2 \leq i \leq 4$. Then $N_2[x_1] \cup N_1[x_2] \cup N_0[x_3] = V(T)$.

**Case 3.** $|S \cap N_2(v)| = 3$.

Suppose $v_4, v_5 \notin S \cap N_2(v)$, this means that $d(v, v_j) \geq 3$,( $j = 4,5$ ). Let $x_1 = v$, $x_2 = v_4$ and $x_3 = v_5$, then $N_2[x_1] \cup N_1[x_2] \cup N_0[x_3] = V(T)$.

**Case 4.** $|S \cap N_2(v)| = 2$.

Suppose $v_3, v_4, v_5 \notin S \cap N_2(v)$, this means that $d(v, v_j) \geq 3$,($j = 3,4,5$). Consider $V(T) - \{v_3, v_4, v_5\} \subset N_2[v]$, the structure of induced subtree $T[S]$ are only six cases, see Figure 2. (a)(b)(c)(d)(e)(f). No matter what case, we always can select $x_1 = v$, $x_2, x_3$ such that $N_2[x_1] \cup N_1[x_2] \cup N_0[x_3] = V(T)$.

**Figure 2**

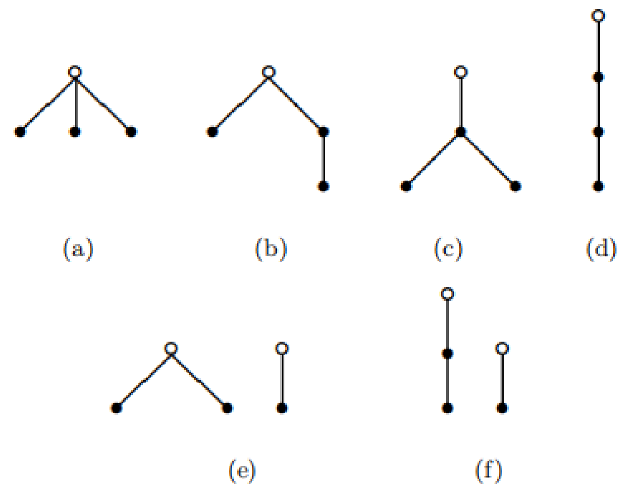


**Figure 2** The Structure of Induced Subtree T[S] (The Black Vertices Are V3, V4, V5 and White Vertices Are V1, V2)

**Case 5.** $|S \cap N_2(v)| = 1$.

Suppose $S \cap N_2(v) = \{v_1\}$ and $d(v, v_j) \geq 3, (j = 2,3,4,5)$. Since $T$ is connected, there is at least one vertex of $v_2, v_3, v_4, v_5$ adjacent to $v_1$, suppose $v_2$ is adjacent to $v_1$. Now we let $x_1 = y$ such that $V(T) - \{v_3, v_4, v_5\} \subset N_2[y]$, then by similar analysis as (e)(f) of case 4, we can select $x_2$, $x_3$ such that $N_2[x_1] \cup N_1[x_2] \cup N_0[x_3] = V(T)$.

As for the other cases $\Delta = n-5, n-4, n-3$, we always similar as case $\Delta = n-6$ to show that there exits $v_1, v_2, v_3 \in V(T)$ such that $N_2[x_1] \cup N_1[x_2] \cup N_0[x_3] = V(T)$. So by Proposition 1.1, we have $b(T) = 3$. Again, by Proposition 1.8, we know $b(T) \geq 3$, so we get $b(T) = 3$.

**Theorem 2.2.** Let $G$ be a connected graph with order $n$ and the maximum degree $\Delta$. If $n-6 \leq \Delta \leq n-3$, then $b(G) = 3$.

Proof. By Proposition 1.8, we have $b(G) \geq 3$. Let $T$ be a spanning tree of graph $G$ satisfies $\Delta(G) = \Delta(T)$, by Theorem 2.1 and Proposition 1.2, we directly get $b(G) = b(T) = 3$. So, we have $b(G) = 3$.

In fact, the condition on the maximum degree in Theorem 2.1 is the best possible. We can show that the condition cannot be relaxed to $n-7 \leq \Delta(G) \leq n-3$. First, we introduce a spider graph $F$ which is illustrated in Figure 3.

**Figure 3**

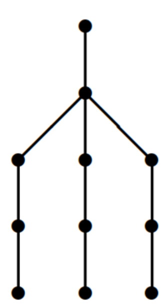

**Figure 3** Spider graph $F$

Note that $SP(3,3)$ is a subtree of $F$ and $F$ is a subtree of $SP(4,3)$, by Proposition 1.7, we get $b(SP(3,3)) = b(SP(4,3)) = 4$. Again, by Proposition 1.4, we directly get $b(F) = 4$. Based on this, the following result is clear.

**Theorem 2.3.** Let $T$ is a tree with order $n$ and maximum degree $\Delta = n - 7$. If $T$ has an induced subgraph $F$, then $b(T) \geq 4$.

Next, we consider the necessary condition on maximum degree and bound the number of edges for $b(G) = 3$, respectively.

**Theorem 2.4.** If the burning number of a connected graph $G$ with order $n$ is $b(G) = 3$, then $\frac{\sqrt{4n-11}-1}{2} \leq \varDelta \leq n-3$.

Proof. It is clear that $\varDelta \leq n-3$. Further, consider $b(G) = 3$, we suppose the degree of the first and the second fire source are both $\Delta$, then we have $(\varDelta^2 - \varDelta + \varDelta + 1) + (\varDelta + 1) + 1 \geq n$. This means we get $n \geq \frac{\sqrt{4n-11}-1}{2}$. This completes the proof.

**Remark 1.** The upper bound of Theorem 2.4 meets the comet graph $C_{4,n-4}$ and the lower bound meets the graph $H$ with order $n$ and $4n - 11$ is the square, $H$ obtained from $\varDelta + 1$ stars $K_{1,\varDelta}$ and $K_1$ by identifying one 1-degree vertex of $\Delta$ stars $K_{1,\Delta}$ together first to get tree $T_1$ and then connecting one 1-degree vertex of $T_1$ and one 1-degree vertex of the last star $K_{1,\varDelta}$ with $K_1$.

**Theorem 2.5.** If the burning number of graphs $G$ with order $n$ is $b(G) = 3$, then $n - 3 \leq |E(G)| \leq \frac{n^2-3n}{2}$.

Proof. If $G$ is connected, then $|E(G)| \geq n - 1 > n - 3$. If $G$ is not connected, consider $b(G) = 3$ and let $G$ has less edges as possible, then $G$ is a forest with three components. So $|E(G)| \geq n - 3$. On the other hand, consider the fact that $b(H) = 2$ if and only if the maximum degree of $H$ is $|H| - 1$ or $|H| - 2$. If $b(G) = 3$, the maximum degree of $G$ is $\Delta \leq n - 3$. This means we can by removing at least 2 edges from each vertex of complete graph $K_n$ to estimate the number of edges in $G$. Thus, we are removing a cycle $C_n$ from $K_n$ to let $\varDelta \leq n - 3$. This implies $|E(G)| \leq \frac{n(n-1)}{2} - n = \frac{n^2-3n}{2}$.

**Remark 2.** The upper bound in Theorem 2.5 meets the comet graph $K_n - C_n$ and the lower bound meets the graph $K_1 \cup K_2 \cup K_{1,n-4}$.

## 3. CONCLUSION

In this paper, we determine the degree condition of a graph $G$ when $b(G) = 3$, and then discuss the bound of the maximum degree and the number of edges in graph $G$ when $b(G) = 3$. However, the sufficient-necessary maximum degree conditions for $b(G) = 3$ are not known, which need further study in future.

## CONFLICT OF INTERESTS

None.

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