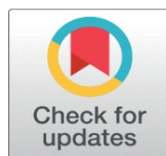# ESTIMATE OF DISK NOT CONTAINING ROOTS OF POLYNOMIAL FUNCTIONS

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ABSTRACT

Suppose we have a polynomial function. Also suppose coefficients of polynomial follow a certain pattern of decreasing or increasing in magnitude. Then we have many results for providing the regions containing all the roots of polynomial functions. Here, in this paper we prove a result that gives a disk or circular region containing no roots of function, thereby our result finally gives annular region containing all roots of polynomial function and hence thereby improves the earlier proved, results.

Keywords: Polynomials, Bounds, Modulus, Disk, Region, Zeros. Roots

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1. INTRODUCTION

If $f(z)$ is a nth degree polynomial function, then by the Fundamental Theorem of Algebra, we know that $f(z)$ has at least one root in complex plane and repeated application of the theorem tells that $f(z)$ has exactly roots in whole Argand plane. But the theorem, however, does not provide any information about the location of roots of a polynomial function. The issue of finding position of the roots of a polynomial function has been of great interest. This could be observed by glancing at the books of Marden (1949), Milovanovic et al. (1994), Sheil-Small (2002) and Rahman and Schmeisser (2002). We have also other recent articles on same area

Aziz and Zargar (1966), Daras and Rassias (2015), Jain (2009), Rahman and Schmeisser (2002), Rassias and Gupta (2016), Shah and Liman (2007), Vieira (2017) on the subject. Since the days of Gauss and Cauchy, many well-known mathematicians have contributed to the further growth of the subject. Here we first mention the following result of Cauchy Aziz and Mohammad (1984) that is commonly popular as Cauchy's Theorem.

Theorem A. If $f(z) = z^n + \sum_{\nu=0}^{n-1} c_\nu z^\nu$ be polynomial function, then all the roots of lie in

$$|z| \le 1 + M,$$

<div align="right">**Equation 1**</div>

were

$$M = \max_{0 \le \nu \le n-1} |c_\nu|$$

Next elegant result that is commonly famous as Enestrom-Kakeya Theorem, and firstly proved by Enstrom Enestrom (1920) and later independently by Kakeya Kakeya (1912) and Hurwitz (1913).

Theorem B. If $f(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ is a polynomial of degree n, such that

$$c_n \ge c_{n-1} \ge \cdots\cdots \ge c_1 \ge c_0 > 0$$

<div align="right">**Equation 2**</div>

then $f(z)$ has no roots in $|z| > 1$

In the same sphere Aziz and Zargar (1966), Aziz and Mohammad (1984), Dichler (1996), Lal et al. (2011), Lal (2019) there exist various generalizations and refinements of Theorem B and other related results.

Joyal et al. (1967) augmented Theorem B for the polynomial function having coefficients having all real values. More precisely, they gave the next result.

Theorem C. If $f(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ is a polynomial function of degree n, with property

$$c_n \ge c_{n-1} \ge \cdots\cdots \ge c_1 \ge c_0$$

<div align="right">**Equation 3**</div>

then all roots of $f(z)$ are contained in

$$|z| \le \frac{c_n - c_0 + |c_0|}{|c_n|}$$

<div align="right">**Equation 4**</div>

If $c_0 > 0$ then this result reduces to Theorem B.

Aziz and Zargar (1966) improved upon the bound in Theorem C.

Theorem D. If $f(z) = \sum\limits_{i=0}^{n} c_i z^i$ is a polynomial function of nth degree, such that for some $M \geq 1$

$$Mc_n \geq c_{n-1} \geq \cdots\cdots \geq c_1 \geq c_0 \qquad \text{Equation 5}$$

then all roots of $f(z)$ are contained in

$$\left| z + M - 1 \right| \leq \frac{1}{\left| c_n \right|} \left\{ Mc_n - c_0 + \left| c_0 \right| \right\} \qquad \text{Equation 6}$$

Rather (1998) augmented the above Theorem D in following.

Theorem E. If $f(z) = \sum\limits_{i=0}^{n} c_i z^i$ is a polynomial function of nth degree, and for some $M \geq 1$

$$Mc_n \geq c_{n-1} \geq \cdots\cdots \geq c_r \leq c_{r-1} \leq \cdots\cdots \leq c_1 \leq c_0, \quad 0 \leq r \leq n-1, \qquad \text{Equation 7}$$

Then $f(z)$ contains all its roots inside

$$\left| z + M - 1 \right| \leq \frac{Mc_n + \left| c_0 \right| - c_0 - 2c_r}{\left| c_n \right|} \qquad \text{Equation 8}$$

2. MAIN THEOREM

The main idea of this paper is to find a region having any roots, inside the disk containing every root. Here, we are able to find such region by proving next result that gives us a root-free region for the polynomial, whose coefficients follow a certain pattern. This result improves upon the result of Enestrom and Kakeya and also some of the other results in this sphere.

Theorem 2. 1.. If $f(z) = \sum\limits_{i=0}^{n} c_i z^i$ be a polynomial function of nth degree, and for some $M \geq 1$ together with $0 \leq r \leq n-1$

$$Mc_n \geq c_{n-1} \geq \cdots\cdots \geq c_r \leq c_{r-1} \leq \cdots\cdots \leq c_1 \leq c_0 \qquad \text{Equation 9}$$

then no roots of $f(z)$ are contained in

$$|z| < \frac{|c_0|}{M(|c_n| + c_n) + c_0 - 2c_r}$$

Equation 10

Remark 2.2. For M=1 the above result improves upon Theorem E due to Rather Rather (1998) and for $K = 1$, $r = 0$ the above result improves upon Theorem C due to Joyal et al. (1967). Furthermore, the result proved here also refines upon the result of Enestrom-Kakeya Enestrom (1920) for $\forall \ c_i > 0, \quad i = 0, 1, 2..., n$.

3. PROOF OF MAIN THEOREM

We prove the main Theorem 2.1 as follows.

Proof of Theorem 2.1.

For the proof of our main theorem, we take a new polynomial function as

$$\begin{aligned} F(z) &= (1-z)f(z) \\ &= c_0 + (c_1 - c_0)z + (c_2 - c_1)z^2 + \cdots\cdots + (c_n - c_{n-1})z^n - c_n z^{n+1} \\ &= c_0 + \sum_{i=1}^{n}(c_i - c_{i-1})z^i - c_n z^{n+1} \end{aligned}$$

or

$$F(z) = c_0 + g(z) \quad \text{say}$$

Equation 11

were

$$g(z) = -c_n z^{n+1} + \sum_{i=1}^{n}(c_i - c_{i-1})z^j$$

Equation 12

From Equation 11

$$|F(z)| \ge |c_0| - |g(z)| \quad \text{for} \qquad |z| = 1$$

Equation 13

Also, for $|z| = 1$, from **Equation 12**, we have

$$\begin{aligned} |g(z)| &\le |c_n| + \sum_{i=1}^{n}|c_i - c_{i-1}| \\ &= |c_n| + |c_n - c_{n-1}| + \sum_{i=1}^{n-1}|c_i - a_{i-1}| \\ &= |c_n| + |Mc_n - c_{n-1} - Mc_n + c_n| + \sum_{i=1}^{r}|c_i - a_{i-1}| + \sum_{i=r+1}^{n-1}|c_i - c_{i-1}| \end{aligned}$$

$$\leq |c_n| + |Mc_n - c_{n-1}| + (M-1)|c_n| + \sum_{i=1}^{r}(c_{-1} - c_i) + \sum_{i=r+1}^{n-1}(c_i - c_{i-1})$$

$$= |c_n| + (M-1)|c_n| + Mc_n - c_{n-1} + c_0 - 2c_r + c_{n-1}$$

$$= M(|c_n| + c_n) + c_0 - 2c_r.$$

Thus for $|z| = 1$,

$$|g(z)| \leq M(|c_n| + c_n) + c_0 - 2c_r$$

Now g (0) =0 and g(z) are analytic, we have, obviously by Schwarz's lemma for $|z| \leq 1$,

$$|g(z)| \leq \{M(|c_n| + c_n) + c_0 - 2c_r\}|z| \qquad \text{Equation 14}$$

Associating the Equation 13 and Equation 14, we get

$$|F(z)| \geq |c_0| - \{M(|c_n| + c_n) + c_0 - 2c_r\}|z|$$
$$> 0$$

if

$$|z| < \frac{|c_0|}{M(|c_n| + c_n) + c_0 - 2c_r}.$$

This implies that $F(z)$ and hence $f(z)$ does not vanish in

$$|z| < \frac{|c_0|}{M(|c_n| + c_n) + c_0 - 2c_r}.$$

Thus, we completed the proof of Theorem 2.1.

4. CONCLUSION

Our result gives a circular region containing no roots or zeros of polynomial functions inside it while other mentioned earlier proved results give circular regions containing all roots. Thus, we have obtained an annular region containing all roots of polynomial the region containing all roots of polynomial function has been reduced in size. So, our result improves the estimate of region having all roots.

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