ISSN: 2454-1907
International Journal of Engineering Technologies and Management Research
April 2021, Vol 8(4), 1 – 5
DOI: https://doi.org/10.29121/ijetmr.v8.i4.2021.903

ON SUPERCONTINUOUS FUNCTIONS IN TOPOLOGY

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DOI: https://doi.org/10.29121/ijetmr.v8.i4.2021.903

Article Citation: Ramesh Bhat. (2021). ON SUPERCONTINUOUS FUNCTIONS IN TOPOLOGY. International Journal of Engineering Technologies and Management Research, 8(4), 1-5. https://doi.org/10.29121/ijetmr.v8.i4.2021.903

Published Date: 12 April 2021

Keywords:
Super Continuous Functions
T3 Spaces
Neighbourhood Functions
ECT

Abstract

The aim of this paper is to introduce and study new classes of continuous functions and its properties in topological spaces comparing with different types of continuous functions.

1. INTRODUCTION

In this paper we studed the basic concepts of super-continuous and their basic results and some other useful results have been studied. Super-continuous maps were first introduced and investigated by B. M. Munshi and D. S. Bassan [1] in 1982. Later J. L. Reilly and M. K. Vamanamoorthi [2] continued the study of super-continuous mappings and obtained many useful results in 1983.

Super continuous functions contained in the class of continuous functions. Munshi and Bassan [1] defined the super-continuous map as follows: A map $f: X \to Y$ is said to be super-continuous at a point $x \in X$ if for every neighbourhood M of f (x) there is a neighbourhood N of x such that $f(\overline{N})^0 \subseteq M$. This class is contained in the class of continuous mappings. Super-continuous mappings turn out to be the natural tool for studying nearly compact space of Singal and Mathur [5], almost regular spaces of Singal and Arya [3] and almost completely regular spaces of Singal and Arya [4], M. K. Signal and A.R. Singal [6] Almost continuous mapping and N. V. Velicko [7], H-closed topological spaces Various properties of such mappings have been discussed.

2. PRILIMINARIES

Throughout this dissertation work (X, τ), (Y, μ) and (Z,η) represent non-empty topological spaces on which no separation axioms are assumed unless explicitly stated, and they are simply written X, Y and Z respectively. For a subset A of (X, τ), the closure of A, the interior of A with respect to τ are denoted by $\overline{A}$ and A^0 respectively. The complement of A is denoted by A^c. Now, we recall some of the following definitions

3. SUPER CONTINUOUS FUNCTIONS

Definition 2.1 : A map f: X $\to$ Y is said to be super-continuous at a point x$\in$X if for every neighbourhood M of f (x) there is a neighbourhood N of x such that f ($\overline{N}$)$^0 \subseteq$ M.

Example 2.2 : Let X={a, b, c, d}, Y={1, 2, 3, 4}, $\Im_1$={X, ϕ, {a, b}, {a, b, d}} and $\Im_2$={ Y, ϕ, {1, 3}, {1, 2, 3}}. Then (X, $\Im_1$) and (Y, $\Im_2$) are topological spaces.

Let f : X $\to$ Y be a map defined as f (a) = f (b) =1, f (c) = 2, f (d) = 3.

f is continuous:

1) f is continuous at a$\in$X, then for an open set {1, 3} containing f (a) =1, there exists an open set {a, b} containing a such that f [{a, b}] = {1} $\subseteq$ {1, 3}.
2) f is continuous at b$\in$X, then for an open set {1, 3} containing f (b) =1, there exists an open set {a, b} containing b such that f [{a, b}] = {1} $\subseteq$ {1, 3}.
3) f is continuous at c$\in$X, then for an open set {1, 2, 3} in Y containing f (c) = 2, there exists an open set X containing c such that f [X] = {1, 2, 3} $\subseteq$ {1, 2, 3}.
4) f is continuous at d$\in$X, then for an open set {1, 2, 3} in Y containing f (d) = 3, there exists an open set X containing c such that f [X] = {1, 2, 3} $\subseteq$ {1, 2, 3}.

Therefore, f is continuous as each point of X.

f is not super - continuous at x = a:

For an open set {1, 3} in Y containing f (a) = 1, there is a neighbourhood {a, b} of a such that f [$\overline{\{a,b\}}$]^0 = f (X) = X $\not\subset$ {1, 3}.

Therefore, f is not super continuous at x = a.

Definition 2.3 : A mapping f: X $\to$ Y is said to be super-continuous [denoted by SC] if it is super-continuous at each point of X.

Definition 2.4 : A set G is said to be δ-open if for each x$\in$G, there exists a regular open set H such that x$\in$ H $\subseteq$ G, or equivalently G can be expressed as arbitrary union of regular open sets.

A set G is δ-closed if and only if its complement is δ-open.

Theorem.2.5: Let f: X $\to$ Y be a map. Then the following are equivalent.

1) f if super continuous.
2) Inverse image of every open subset of Y is a δ-open subset of X.
3) Inverse image of every closed subset of Y is a δ-closed subset of X.
4) For each point x of X and for each open neighbourhood M of f(x), there is a δ-open neighbourhood N of x such that f(N) $\subset$ M.

Proof: (a) $\Rightarrow$ (b): Suppose (a) holds.

Let U be any open subset of Y and let x$\in$ f $^{-1}$(U). Then f (x) $\in$U. Since f is super continuous, from (a), there exists an open set V in X such that x$\in$V and f ($\overline{V}^0$) $\subset$ U. Thus x$\in$ $\overline{V}^0$ $\subset$ f $^{-1}$(U). Therefore f^{-1}(U) is expressible as an arbitrary union of regularly open sets. Hence f $^{-1}$(U) is δ-open.

(b) $\Rightarrow$ (c): Let U be a closed set in Y. Then Y – U is open set of Y. Then from (b), f $^{-1}$(Y – U) is δ-open subset of X. Therefore f $^{-1}$(Y – U) = X – f $^{-1}$(U) is δ-open subset of X. Hence f $^{-1}$(U) is δ-closed set of X.

(c) $\Rightarrow$ (d): Let M be an open set in Y containing f (x), that is, f (x) $\in$ M. Since Y – M is closed, by (c), f $^{-1}$(Y – M) is δ-closed subset of X. Therefore f $^{-1}$(M) is δ-open. Also x $\in$ f^{-1}(M). Let N = f^{-1}(M). Then N is a δ-open neighbourhood of x such that f (N) $\subseteq$ M.

(d) $\Rightarrow$ (a): Let for each x$\in$X and for each neighbourhood M of f (x) there is a neighbourhood N of f(x), so N is δ-open neighbourhood N of x such that f (N) $\subseteq$ M, from (d). Then f ($\overline{N}^0$) $\subseteq$ M. So f is super continuous.

Hence the proof.

Definition 2.6 : A space X is said to be semi-regular if for each point x of the space and each open set U containing x there is a open set V such that $x \in V \subset \overline{V^0} \subset U$.

Theorem 2.7: Let $f : X \to Y$ be a continuous mapping of a semi-regular space X into Y. Then f is super - continuous.

Proof: Let $x \in X$ and let G be an open set containing f (x). Since f is continuous, $f^{-1}(G)$ is open in X. Since X is semi-regular space, there is an open subset M of x such that $x \in M \subset \overline{M^0} \subseteq f^{-1}(G)$. Therefore $f(x) \in f(M) \subseteq f(\overline{M^0}) \subseteq G$. That is, $f(\overline{M^0}) \subseteq G$. Hence f is super continuous.

Remark 2.8 : Every open set in a T_3 -space can be written as the union of regular open sets.

Corollary 2.9 : Let X be a T_3 topological space and let $f : X \to Y$ be a continuous, then f is super-continuous.

Proof: Proof follows from every regular space (or T_3) space is semi regular.

Theorem 2.10: Let X and Y are topological spaces. Then a mapping $f : X \to Y$ is super-continuous if and only if the inverse image under f of every member of a base (sub base) for Y is δ-open in X.

Proof: Let f be super continuous and B be a subbase for Y. Since each member of B* is open in Y, it follows that from the Theorem that $f^{-1}(Y)$ is δ-open for every $Y \in B$.

Conversely, let $f^{-1}(Y)$ be a δ-open in X for every $Y \in B$ and let H be any open set in Y. Let β be a family of all finite intersections of members of B_* so B is a base for Y. If $B \in \beta$, then there exists $\nu_1, \nu_2 \ldots\ldots\ldots \nu_n$ (n is finite) in B_* such that $B = \nu_1 \cap \nu_2 \cap \ldots\ldots\ldots \cap \nu_n$. Then $f^{-1}(B) = f^{-1}(\nu_1) \cap f^{-1}(\nu_2) \cap \ldots\ldots\ldots \cap f^{-1}(\nu_n)$. By hypothesis, each $f^{-1}(\nu_i)$ is δ-open in X and therefore $f^{-1}(B)$ is also δ-open in X, since β is a base for X. $H = \cup\{B: B \in C \subset \beta\}$. Then $f^{-1}(H) = f^{-1}[\cup\{B:B \in C \subset \beta\}] = \cup\{f^{-1}(B): B \in C\}$ which is δ-open in X, since $f^{-1}(B)$ is δ-open in X. Thus $f^{-1}(H)$ is δ-open in X for every open set H in Y and therefore f is super continuous.

Definition 2.11 : A point is said to be a δ-adherent point of a set P in a space X if equivalently, every regular open set containing x has non-empty intersection with P.the interior of every closed neighbourhood of the point x intersects P or

Definition 2.12 : The set $(P)_\delta$ of all δ-adherent point of a set P is called the δ-closure of the set P.

Theorem 2.13 : A mapping f from a space X into another space Y is super -continuous if and only if another space Y is super -continuous if and only if $f(A)_\delta \subset \overline{f(A)}$ for every subset A of X.

Proof: Let f be a super continuous. Since $\overline{f(A)}$ is closed in Y, then by $f^{-1}(\overline{f(A)})$ is δ-closed in X, since f is super -continuous. Now $f(A) \subset \overline{f(A)}$ implies, $A \subset f^{-1}(\overline{f(A)})$. Therefore $(A)_\delta \subset [f^{-1}(\overline{f(A)})]_\delta = f^{-1}(\overline{f(A)})$. Therefore $f(A)_\delta \subset f[f^{-1}(\overline{f(A)})] \subset \overline{f(A)}$. So $f(A)_\delta \subset \overline{f(A)}$.

Conversely, let $f(A)_\delta \subset \overline{f(A)}$ for every subset A of X. Let F be any closed set in Y so that $\overline{F} = F$. Now $f^{-1}(\overline{F})$ is a subset of X implies that $f[f^{-1}(F)]_\delta \subset \overline{f[f^{-1}(F)]} \subset \overline{F} = F$ implies $[f^{-1}(F)]_\delta \subset f^{-1}(F)$. Therefore $[f^{-1}(F)]_\delta = f^{-1}(F)$, so $f^{-1}(F)$ is δ-closed set in X. Hence f is super -continuous.

Theorem 2.14 : A mapping f from a space X into another space Y is super-continuous if and only if $[f^{-1}(B)]_\delta \subset f^{-1}(\overline{B})$ for every $B \subset Y$.

Proof: Let f be super-continuous. Since $\overline{B}$ is closed in Y and since f is super-continuous, $f^{-1}(\overline{B})$ is δ-closed set in X. Therefore $f^{-1}(\overline{B}) = [f^{-1}(\overline{B})]_\delta$. Now $B \subset \overline{B} \subset [(\overline{B})]_\delta$ implies $[f^{-1}(B)]_\delta \subset [f^{-1}(\overline{B})]_\delta$ implies $[f^{-1}(B)]_\delta \subset f^{-1}(\overline{B})$.

Conversely, let the condition hold and let F be any closed set in Y. Therefore $\overline{F} = F$. Now $[f^{-1}(F)]_\delta \subset f^{-1}(\overline{F}) = f^{-1}(F)$. But $f^{-1}(F) \subset \overleftrightarrow{f^{-1}(F)} \subset [f^{-1}(F)]_\delta$. Hence $f^{-1}(F) = [f^{-1}(F)]_\delta$. Therefore $f^{-1}(F)$ is δ-closed set in X. Hence f is super-continuous.

Definition 2.15: A point x is called a δ-adherent point of a filter base Ŧ if and only if $x \in \cap\{[F]_\delta : F \in Ŧ\}$.

Definition 2.16: A filter base is said to be δ-coverage of a point x (written as $Ŧ^\delta \to x$) if every regular open set containing x contains $F \in Ŧ$.

Theorem 2.17 : Let $f : X \to Y$ be a mapping. Then f is super-continuous at $x \in X$ if and only if the filter base $f(U(x)) \to f(x)$, where U

Theorem 2.18 : A mapping f: $X \to Y$ is super continuous on X if and only if f (U) $\to$ f (x) for each x$\in$X and each filter base U that δ-converges to x.

Proof: Assume that f is super continuous on X and let U $\xrightarrow{\delta}$ x. Let W be a neighbourhood of f (x). Then x$\in$ f^{-1}(W) and f^{-1}(W) is δ-open since f is super- continuous. Therefore x$\in$H such that f (H) $\subset$ W where H is regular open and H $\subset f^{-1}$(W). Therefore, there exits a u$\in$ U such that u$\in$H. Therefore f (u) $\subset$ f (H) $\subset$ W. Therefore f (U) $\to$ f (x).

Conversely, let W be any open subset of Y containing f (x). Let B be any subset of X. We have to prove that f ([B]$_\delta$) $\subset \overline{f(B)}$.

Let b $\in$[B]$_\delta$. Let U be a filter base on B with U-δ-covering to b so that f (U) $\to$ f (b). Since f (U) is a filter base on f (B), therefore f (b)$\in$ [f (B)] $\subset \overline{f(B)}$ Therefore f is super continuous.

Theorem 2.19 (Restricting the range) : If f : $X \to Y$ is super-continuous and f (X) is taken with the subspace topology then f : $X \to$ f (X) is super-continuous.

Proof : Let f : $X \to Y$ be super-continuous. Let U be an open subset of Y then f^{-1}(U) is δ-open in X. Now f^{-1}(U$\cap$ f (X)) = f^{-1}(U) $\cap$ f^{-1}[f (X))] = f^{-1}(U) $\cap$X = f^{-1}(U) is δ-open. Therefore f : $X \to Y$ is super-continuous.

Theorem 2.20 (Expanding the range) : Let f : $X \to Y$ is super-continuous. If Z is a space having Y as a subspace then the function h : $X \to Z$ obtained by expanding the range of f is super-continuous.

Proof: We have to show that h : $X \to Z$ is super-continuous. As Z has Y as a subspace, h is the composite of the map f : $X \to Y$ which is super-continuous and the inclusion map g : $Y \to Z$ which is continuous. Thus, h is super-continuous.

Definition 2.21 : A mapping f : $X \to Y$ is said to be almost open if the image of every regularly open subset of X is an open subset of Y.

A mapping f : $X \to Y$ is said to be almost closed if the image of every regularly closed subset of X is a closed subset of Y.

Definition 2.22: A mapping f : $X \to Y$ is said to be almost continuous at a point x$\in$X if for every neighbourhood M of f (x) there is a neighbourhood N of x such that f (N) $\subset \overline{M^0}$.

Theorem 2.23 : If f is an almost open, super-continuous mapping X onto Y and if g is a mapping of Y into Z then gof is super-continuous if and only if g is continuous.

Proof : Let g be continuous. Let G be an open set in Z. Then g^{-1} (G) is open in Y. Also f is super-continuous, f^{-1}(g^{-1}(G)) is δ-open in X. Therefore f^{-1}(g^{-1}(G)) = (g$\circ$ f)$^{-1}$(G) is δ-open in X. Hence gof is super-continuous.

Conversely, let gof be super-continuous. Let G be an open subset of Z. Therefore (gof)$^{-1}$ (G) is δ-open subset of X since gof is super-continuous. That is f^{-1}(g^{-1}(G)) is a δ-open subset in X. Also, f is almost open and onto, f [f^{-1} (g^{-1}(G))] = g^{-1}(G) is open in Y. Hence g is continuous.

Theorem 2.24 : Let X, Y, Z be topological spaces and the mapping f : $X \to Y$ be almost continuous and g : $Y \to Z$ be super-continuous. Then the composition gof : $X \to Z$ is continuous.

But if f : $X \to Y$ is almost continuous and g$\circ$ f : $X \to Z$ is continuous, then g : $Y \to Z$ need not be super-continuous.

Example 2.25 : Let (R, $\Im_1$) be the topological space where $\Im_1$ is the topology consisting of ϕ, R and complements of countable subsets of R. Let X = { a, b } and $\Im_2$ = {X, ϕ, {a} }. Let f : $R \to X$ be defined as follows:

$$f(x) \begin{cases} a \text{ if x is irrational} \\ b \text{ if x is rational} \end{cases}$$

Let Y = {1, 2} and $\Im_3$ = {Y, ϕ, {2}}. Let g : $X \to Y$ be defined as g (a) = 2, g (b) = 1. Then f : $R \to X$ is almost continuous and gof : $R \to Y$ is continuous but g : $X \to Y$ is not super-continuous.

SOURCES OF FUNDING

This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

CONFLICT OF INTEREST

The author have declared that no competing interests exist.

ACKNOWLEDGMENT

None.

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