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THEKERNEL OF N- DIMENSIONAL FRACTIONAL FOURIER TRANSFORM

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Abstract:
In this paper we have developed the kernel of N-dimensional fractional Fourier transform by extending the definition of first dimensional fractional Fourier transform. The properties of kernel up to N- dimensional are also presented here which is missing in the literature of fractional Fourier transform. The properties of kernel of fractional Fourier transforms up to N- dimensional will help the researcher to extend their research in this aspect.

Keywords: Fourier Transform; Fractional Fourier Transform; N-Dimensional Fractional Fourier Transform; Kernel of N- Dimensional Fractional Fourier Transform.

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1. Introduction

The idea of fractional operator of Fourier transform (FT) was introduced by V. Namias in 1980 [4]. In which he had descripted first time the comprehensive definition and mathematical frame work of Fractional Fourier Transform (FRFT).

The Fractional Fourier transform (FRFT) depends on a parameter α that is associated with the angle in phase plane. This leads to the generalization of notion of space (or time) and frequency domain which are central concepts of signal processing. The kernel of the fractional FT is except for a phase factor, equal to the propagator of the non-stationary Schrodinger equation for the harmonic Oscillator, this transform is also used in optics [10,11]. FRFT was first introduced as a way to solve certain classes of ordinary and partial differential equations arising in quantum mechanics [4]. FRFT has found applications in areas of signal processing such as repeated altering, fractional convolution and correlation, beam forming, optional filter, convolution, filtering and wavelet transforms, time frequency representation [7]. The FRFT is basically a time-frequency distribution. It provides us with an additional degree of freedom (order of the transform), which is in most cases results in significant gain over the classical Fourier transform. With the development of FRFT and related concepts, we see that the ordinary frequency domain is merely a special case of a continuum of fractional Fourier domains. Every property and application of the ordinary Fourier transform becomes a special case of the FRFT. So, in every area in which Fourier

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transforms and frequency domain concepts are used, there exists the potential for improvement by using the FRFT [3,5,9]

2. Results and Discussions

In this section we will consider the definition of fractional Fourier transform as one, two and three dimensional along with properties of kernel and their proof and will extend this concept to n-dimensional fractional Fourier transform.

2.1. 1-Dimensional Fractional Fourier Transform

The operator of FRFT has an order parameter α that is an arbitrary angle $\alpha\frac{\pi}{2}$, and it will reduce to Fourier transform whenever the rotation of angle α is $\frac{\pi}{2}$ a detailed discussion is found in [1-4] The one-dimensional fractional Fourier transform is defined as

$$\mathcal{F}^{\alpha}[f(x_1)] = \int_{-\infty}^{\infty} k^{\alpha}(x_1,u_1)f(x_1)dx_1 \tag{1}$$

The $k^{\alpha}(x_1,u_1)$ is the kernel and defined as

$$k^{\alpha}(x_1,u_1) = \begin{cases} a(\alpha)e^{b(\alpha)\left(\left(x_1{}^2+u_1{}^2\right)\cos\alpha-2u_1x_1\right)}, & if\alpha \neq n\pi \\ \delta(x_1-u_1), & if\alpha = 2n\pi \\ \delta(x_1+u_1), & if\alpha = (2n\pm 1)n\pi \end{cases} \tag{2}$$

$$\text{where} \quad \begin{cases} a(\alpha) = \sqrt{\frac{1-jcot\alpha}{2\pi}} \\ b(\alpha) = \frac{j}{2\sin\alpha} \end{cases} \tag{3}$$

2.2. Properties of kernel

Following are the properties of 1-dimensional fractional Fourier transform kernel

$$\begin{cases} k^{\alpha}(x_1,u_1) = k^{\alpha}(u_1,x_1) \\ k^{-\alpha}(x_1,u_1) = k^{*\alpha}(x_1,u_1) \\ k^{\alpha}(-x_1,u_1) = k^{\alpha}(x_1,-u_1) \\ \int_{-\infty}^{\infty} k^{\alpha}(x_1,u_1)k^{\beta}(u_1,z_1)du_1 = k^{\alpha+\beta}(x_1,u_1) \\ \int_{-\infty}^{\infty} k^{\alpha}(x_1,u_1)k^{*\alpha}(x_1,u'_1)du_1 = k^{\alpha}(u_1,u'_1) \end{cases} \tag{4}$$

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The proofs of these properties given in (4) are discussed by Almaida in [4]

2.3. 2-Dimensional Fractional Fourier Transform

The 2-dimensional fractional Fourier transform is also the generalization of 2-dimensional Fourier transform several properties of (2-D) FRFT have been developed by generalizing the properties of the ordinary (2-D) Fourier Transform (FT). The two dimensional fractional Fourier transform with parameter α of $f(x_1, x_2)$ denoted by $\mathcal{F}^{\alpha_1,\alpha_2}\{f(x_1, x_2)\}$ performs a linear operation given by the integral transform [12,14,16,18]

$$\mathcal{F}^{\alpha_1,\alpha_2}\{f(x_1, x_2)\}(u_1, u_2) = \mathcal{F}^{\alpha_1,\alpha_2}(u_1, u_2)$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2) K^{\alpha_1,\alpha_2}(x_1, x_2; u_1, u_2)\ dx_1 dx_2 \tag{5}$$

Where $K^{\alpha_1,\alpha_2}(x_1, x_2; u_1, u_2) = K^{\alpha_1}(x_1, u_1) K^{\alpha_2}(x_2, u_2)$ and defined as

$$K^{\alpha_1,\alpha_2}(x_1, x_2; u_1, u_2)$$
$$= a(\alpha) e^{b(\alpha)\left(({x_1}^2 + {u_1}^2 + {x_2}^2 + {u_2}^2)\cos\alpha - 2(x_1 u_1 + x_2 u_2)\right)} \tag{6}$$

In case of the two-dimensional FRFT there are two angles of rotation expressed as $\alpha_1 = m\frac{\pi}{2}$, $\alpha_2 = m\frac{\pi}{2}$if one of these angle is zero, the 2-D FRFT reduced to 1-D FRFT. The properties of 2-D transformation kernel are defined in [15] as

$$\begin{cases} k^{\alpha_1,\alpha_2}(x_1, x_2; u_1, u_2) = k^{\alpha_1,\alpha_2}(u_1, u_2; x_1, x_2) \\ k^{-\alpha_1,-\alpha_2}(x_1, x_2; u_1, u_2) = k^{*\alpha_1,\alpha_2}(x_1, x_2; u_1, u_2) \\ k^{\alpha_1,\alpha_2}(-x_1, -x_2; u_1, u_2) = k^{\alpha_1,\alpha_2}(x_1, x_2; -u_1, -u_2) \\ k^{\alpha_1,\alpha_2}(x_1, x_2; u_1, u_2) = k^{\alpha_1}(x_1, u_1) k^{\alpha_2}(x_2, u_2) \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k^{\alpha_1,\alpha_2}(x_1, x_2; u_1, u_2) k^{\beta_1,\beta_2}(u_1, u_2; z_1, z_2) du_1\, du_2 = k^{\alpha_1+\beta_1,\alpha_2+\beta_2}(x_1, x_2; u_1, u_2) \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k^{\alpha_1,\alpha_2}(x_1, x_2; u_1, u_2) k^{*\alpha_1,\alpha_2}(x_1, x_2; u'_1, u'_2) du_1\, du_2 = k^{\alpha_1,\alpha_2}(u_1, u_2, u'_1, u'_2) \end{cases} \tag{7}$$

2.4. Three-Dimensional Fractional Fourier Transform

In a similar manner the 3-D FRFT can be defined as

$$\mathcal{F}^{\alpha_1,\alpha_2,\alpha_3}\{f(x_1, x_2, x_3)\} = \mathcal{F}^{\alpha_1,\alpha_2,\alpha_3}(u_1, u_2, u_3)$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2) K^{\alpha_1,\alpha_2,\alpha_3}(x_1, x_2, x_3; u_1, u_2, u_3)\ dx_1 dx_2\ dx_3 \tag{8}$$

Where $K^{\alpha_1,\alpha_2,\alpha_3}(x_1, x_2, x_3; u_1, u_2, u_3)$ defined as

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$$K^{\alpha_1,\alpha_2,\alpha_3}(x_1,x_2,x_3;u_1,u_2,u_3) = a(\alpha)e^{b(\alpha)\left(\left(x_1{}^2+u_1{}^2+x_2{}^2+u_2{}^2+x_3{}^2+u_3{}^2\right)\cos\alpha-2(x_1u_1+x_2u_2+x_3u_3)\right)} \tag{9}$$

In case of the 3-D FRFT there are three angles of rotation expressed as $\alpha_1 = n_1\frac{\pi}{2}$, $\alpha_2 = n_2\frac{\pi}{2}$, $\alpha_3 = n_3\frac{\pi}{2}$if one of or two of these angles are zero, the 3-D FRFT reduced to 2-D FRFT and 1-D FRFT. The properties of 3-D transformation kernel are defined as

$$\begin{cases} k^{\alpha_1,\alpha_2,\alpha_3}(x_1,x_2,x_3;u_1,u_2,u_3) = k^{\alpha_1,\alpha_2,\alpha_3}(u_1,u_2,u_3;x_1,x_2,x_3) \\ k^{-\alpha_1,-\alpha_2,-\alpha_3}(x_1,x_2,x_3;u_1,u_2,u_3) = k^{*\alpha_1,\alpha_2,\alpha_3}(x_1,x_2,x_3;u_1,u_2,u_3) \\ k^{\alpha_1,\alpha_2,\alpha_3}(-x_1,-x_2,-x_3;u_1,u_2,u_3) = k^{\alpha_1,\alpha_2,\alpha_3}(x_1,x_2,x_3;-u_1,-u_2,-u_3) \\ k^{\alpha_1,\alpha_2,\alpha_3}(x_1,x_2,x_3;u_1,u_2,u_3) = k^{\alpha_1}(x_1,u_1)k^{\alpha_2}(x_2,u_2)k^{\alpha_3}(x_3,u_3) \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k^{\alpha_1,\alpha_2,\alpha_3}(x_1,x_2,x_3;u_1,u_2)k^{\beta_1,\beta_2}(u_1,u_2;z_1,z_2)du_1\,du_2du_3 \\ \qquad = k^{\alpha_1+\beta_1,\alpha_2+\beta_2}(x_1,x_2,x_3;u_1,u_2) \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k^{\alpha_1,\alpha_2,\alpha_3}(x_1,x_2,x_3;u_1,u_2,u_3)k^{*\alpha_1,\alpha_2,\alpha_3}(x_1,x_2,x_3;u'_1,u'_2,u'_3)du_1\,du_2du_3 \\ \qquad = k^{\alpha_1,\alpha_2,\alpha_3}(u_1,u_2,u_3,u'_1,u'_2,u'_3) \end{cases} \tag{10}$$

The kernel properties are given in (7) and (10) can easily be proved by extending the concept of (4).

2.5. *N*-Dimensional Fractional Fourier Transform

Extending the concept of 1-D, 2-D, and 3-D FRFT, the fractional Fourier transform can be extended up to n-dimensional The$n$ dimensional fractional Fourier transform with parameter α of is also defined as

$$K^{\alpha_1,\alpha_2,\alpha_3,\dots,\alpha_n}(x_1,x_2,x_3,\dots,x_n;u_1,u_2,u_3,\dots,u_n) = a(\alpha)e^{b(\alpha)\left(\left(x_1{}^2+u_1{}^2+x_2{}^2+u_2{}^2+\cdots+x_n{}^2+u_n{}^2\right)\cos\alpha-2(x_1u_1+x_2u_2+\cdots+x_nu_n)\right)} \tag{11}$$

Where $\alpha_1 = n_1\frac{\pi}{2}$, $\alpha_2 = n_2\frac{\pi}{2}$, … , $\alpha_n = n_n\frac{\pi}{2}$. In a similar manner the properties of n -D transformation kernel are defined as

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$$\begin{cases} k^{\alpha_1,\alpha_2,\alpha_3,\dots,\alpha_n}(x_1,x_2,x_3,\dots,x_n;u_1,u_2,u_3,\dots,u_n) = k^{\alpha_1,\alpha_2,\alpha_3,\dots,\alpha_n}(u_1,u_2,u_3,\dots,u_n;x_1,x_2,x_3,\dots,x_n) \\ k^{-\alpha_1,-\alpha_2,\dots,-\alpha_n}(x_1,x_2,x_3,\dots,x_n;u_1,u_2,u_3,\dots,u_n) = k^{*\alpha_1,\alpha_2,\alpha_3,\dots,\alpha_n}(x_1,x_2,x_3,\dots,x_n;u_1,u_2,u_3,\dots,u_n) \\ k^{\alpha_1,\alpha_2,\alpha_3,\dots,\alpha_n}(-x_1,-x_2,-x_3,..,-x_n;u_1,u_2,u_3,\dots,u_n) = k^{\alpha_1,\alpha_2,\alpha_3,\dots,\alpha_n}(x_1,x_2,x_3,\dots,x_n;-u_1,-u_2,-u_3,\dots,-u_n) \\ k^{\alpha_1,\alpha_2,\alpha_3,\dots,\alpha_n}(x_1,x_2,x_3,\dots,x_n;u_1,u_2,u_3,\dots,u_n) = k^{\alpha_1}(x_1,u_1)k^{\alpha_2}(x_2,u_2)\dots\ k^{\alpha_n}(x_n,u_n) \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} k^{\alpha_1,\alpha_2,\alpha_3,\dots,\alpha_n}(x_1,x_2,x_3,\dots,x_n;u_1,u_2,u_3,\dots,u_n)k^{\beta_1,\beta_2,\dots,\beta_n}(u_1,u_2,\dots,u_n;z_1,z_2,\dots,z_n,)du_1\,du_2\dots du_n \; (7) \\ \qquad = k^{\alpha_1+\beta_1,\alpha_2+\beta_2,\dots,\alpha_n+\beta_n}(x_1,x_2,x_3,\dots,x_n;u_1,u_2,u_3,\dots,u_n) \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} k^{\alpha_1,\alpha_2,\dots,\alpha_n}(x_1,x_2,x_3,\dots,x_n;u_1,u_2,u_3)k^{*\alpha_1,\alpha_2,\dots,\alpha_n}(x_1,x_2,\dots,x_n;u'_1,u'_2,\dots,u'_n)du_1\,du_2\dots du_n \\ \qquad = k^{\alpha_1,\alpha_2,\dots,\alpha_3}(u_1,u_2,\dots,u_n,u'_1,u'_2,\dots,u'_n) \end{cases} \tag{12}$$

The proof first three properties are very simple. here we are giving proof of 5th property of (12) Which can be written as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} k^{\alpha_1,\alpha_2,\alpha_3,\dots,\alpha_n}(x_1,x_2,x_3,\dots,x_n;u_1,u_2,u_3,\dots,u_n)k^{\beta_1,\beta_2,\dots,\beta_n}(u_1,u_2,\dots,u_n;z_1,z_2,\dots,z_n,)du_1\,du_2\dots du_n$$
$$= \int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\prod_{i=1}^{N} k^{\alpha_i}(x_i,u_i)k^{\beta_i}(x_i,z_i)\,du_i$$

Since from (4) we have

$\int_{-\infty}^{\infty} k^{\alpha}(x_1,u_1)k^{\beta}(u_1,z_1)du_1 = k^{\alpha+\beta}(x_1,u_1)$ which is proved by Almaida in [4]

We can write

$$\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\prod_{i=1}^{N} k^{\alpha_i}(x_i,u_i)k^{\beta_i}(x_i,z_i)\,du_i = \prod_{i=1}^{N} k^{\alpha_i+\beta_i} = k^{\alpha_1+\beta_1,\alpha_2+\beta_2,\dots,\alpha_n+\beta_n}(x_1,x_2,x_3,\dots,x_n;u_1,u_2,u_3,\dots,u_n)$$

Hence, we have the result. Similarly, other properties can be proved.

3. Conclusion

We successfully established the properties of kernels up to N-dimensional fractional Fourier transform which will helpful to extend all the properties of 1-dimensional fractional Fourier transform up to N-dimensional fractional Fourier

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