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A NOTE ON QUADRATIC RICCATI DIFFERENTIAL EQUATION USING PADÉ APPROXIMANT

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ABSTRACT

Semi-analytical methods for solving non-linear models require an initial approach to determine the solutions sought and the calculation of one or more fitting parameters. When the initial approach is chosen correctly, the results can be very precise, but not there is a general method for choosing such an initial approach. In this paper, it is suggested to use directly the serial solution of a non-linear model to find Padé's approximation with highly efficient results.

1. INTRODUCTION

The resolution of non-linear differential equations is a very important problem in the sciences in general since many phenomena are modeled using this type of equation. It is also true that in most cases, it is not possible to find analytical solutions to such models and therefore knowledge of efficient numerical methods to approximate them is essential. Thus, there are several semi-analytical methods that allow us to approximate the solutions numerically, such as the Adomian decomposition method, the differential transformation and the Padé method [1].

For this reason, the Padé Method is widely used in computer calculations. This method has proven to be very useful in obtaining quantitative information about the solution of many interesting problems in physics-mathematics and engineering. The applications of Padé's main approaches are divided into two classes:

- The provision of efficient rational approaches to special mathematical functions
- The acquisition of quantitative information about a function for which you only have qualitative information and coefficients in power series.

The Padé approximations, obtained as the quotient of two polynomials from Taylor's coefficients of series expansion, are the basis of many non-linear methods and have close connections with the famous ε-algorithm, continuous fractions and orthogonal polynomials. The Padé approximations are the non-linear counterpart of the first-order Taylor series expansions used in linear methods.

There are several methods to find Padé's approximations, several of them based on continuous fractions, so special attention is given to these fractions and their most important properties are illustrated.

2. PADÉ APPROXIMANT METHOD

2.1. GENERALITIES OF THE PROPOSED METHOD

The divergence of a power series is indicative of the presence of singularities. This divergence reflects then the inability of the polynomials to approximate a function around that singularity. The basic idea in sum theory is to represent $f(z)$, any function by a convergent expression. One of the methods used, which requires as input only a finite number of terms from a divergent series, is the sums of Padé [2].

In the technique proposed by Padé, the series of powers

$$f(z)=\sum_{i=0}^{\infty} c_i z^i \tag{1}$$

is replaced by a succession of rational functions of the form:

$$R_N^M(z)=\frac{\sum_{i=0}^{M} a_i z^i}{\sum_{i=0}^{N} b_i z^i} \tag{2}$$

Usually we take the standardization $b_0=1$ and the remaining $M+N+1$ coefficients $a_0,\cdots,a_M,b_{1,\dots,}b_N$ in (2), are chosen in such a way that that match the $M+N+1$ coefficients in the power series expansion (1). That is,

$$f(z)-R_N^M(z)=\mathcal{O}(z^{M+N+1}) \tag{3}$$

From where

$$f(z)\sum_{i=0}^{M} b_i z^i-\sum_{i=0}^{M} a_i z^i=\mathcal{O}(z^{M+N+1}) \tag{4}$$

If $\sum a_i z^i$ is a serial representation of $f(z)$, then in many cases $R_N^M(z)\to f(z)$ when $M,N\to\infty$, even if $\sum a_i z^i$ is divergent. Generally, we consider the convergent succession $R_0^J, R_1^{1+J},\cdots$, where $M=N+J$ with J fixed and $N\to\infty$.

The $M+N+1$ coefficients are calculated as follows [3]:

Case 1: $M\geq N$

Developing the expression (4) we have:

$$\begin{array}{rrrrrrr}
c_0+ & c_1z+ & c_2z^2+\ldots+ & c_Nz^N+\ldots+ & c_Mz^M+\ldots+ & c_{M+N}z^{M+N} \\
 & +c_0b_1z+ & c_1b_1z^2+\ldots+ & c_{N-1}b_1z^N+\ldots+ & c_{M-1}b_1z^M+\ldots+ & c_{M+N-1}b_1z^{M+N} \\
 & & +c_0b_2z^2+\ldots+ & c_{N-2}b_2z^N+\ldots+ & c_{M-2}b_2z^M+\ldots+ & c_{M+N-2}b_2z^{M+N} \\
 & & & \vdots & \vdots & \\
 & & + & c_0b_Nz^N+\ldots+ & c_{M-N}b_Nz^M+\ldots+ & c_Mb_Nz^{M+N} \\
=a_0+ & a_1z+ & a_2z^2+\ldots+ & a_Nz^N+\ldots+ & a_Mz^M &
\end{array}$$

By equalizing coefficients in the above expression, we obtain the following systems:

$$\begin{cases} c_M b_1 + c_{M-1} b_2 + \ldots + c_{M-N+1} b_N = -c_{M+1} \\ c_{M+1} b_1 + c_M b_2 + \ldots + c_{M-N+2} b_N = -c_{M+2} \\ \vdots \\ c_{M+N-1} b_1 + c_{M+N-2} b_2 + \ldots + c_M b_N = -c_{M+N} \end{cases} \tag{5}$$

$$\begin{cases} a_0 = c_0 \\ a_1 = c_1 + c_0 b_1 \\ a_2 = c_2 + c_1 b_1 + c_0 b_2 \\ \vdots \\ a_N = c_N + c_{N-1} b_1 + c_{N-2} b_2 + \ldots + c_0 b_N \\ \vdots \\ a_M = c_M + c_{M-1} b_1 + c_{M-2} b_2 + \ldots + c_{M-N} b_N \end{cases} \tag{6}$$

Case 2: $M < N$

$$\begin{aligned} & c_0 + c_1 z + c_2 z^2 + \ldots + c_M z^M + \ldots + c_N z^N + \ldots + c_{M+N} z^{M+N} \\ & + c_0 b_1 z + c_1 b_1 z^2 + \ldots + c_{M-1} b_1 z^M + \ldots + c_{N-1} b_1 z^N + \ldots + c_{M+N-1} b_1 z^{M+N} \\ & + c_0 b_2 z^2 + \ldots + c_{M-2} b_2 z^M + \ldots + c_{N-2} b_2 z^N + \ldots + c_{M+N-2} b_2 z^{M+N} \\ & \vdots \\ & + c_0 b_M z^M + \ldots + c_{N-M} b_M z^N + \ldots + c_N b_M z^{M+N} \\ & \vdots \\ & + c_0 b_N z^N + \ldots + c_M b_N z^{M+N} \\ & = a_0 + a_1 z + a_2 z^2 + \ldots + a_M z^M \end{aligned}$$

and after matching coefficients we get:

$$\begin{cases} c_M b_1 + c_{M-1} b_2 + \ldots + c_0 b_{M+1} = -c_{M+1} \\ c_{M+1} b_1 + c_M b_2 + \ldots + c_1 b_{M+1} + c_0 b_{M+2} = -c_{M+2} \\ \vdots \\ c_{N-1} b_1 + c_{N-2} b_2 + \ldots + c_{N-M-1} b_{M+1} + c_{N-M-2} b_{M+2} + \ldots + c_0 b_N = -c_N \\ \vdots \\ c_{M+N-1} b_1 + c_{M+N-2} b_2 + \ldots + c_{N-1} b_{M+1} + c_{N-2} b_{M+2} + \ldots + c_M b_N = -c_{M+N} \end{cases} \tag{7}$$

$$\begin{cases} a_0 = c_0 \\ a_1 = c_1 + c_0 b_1 \\ a_2 = c_2 + c_1 b_1 + c_0 b_2 \\ \vdots \\ a_M = c_M + c_{M-1} b_1 + c_{M-2} b_2 + \ldots + c_0 b_M \end{cases} \tag{8}$$

By solving systems (5) and (7), we find the coefficients in the denominator of (2):

$$\mathcal{A}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix} = -\begin{bmatrix} c_{M+1} \\ c_{M+2} \\ \vdots \\ c_{M+N} \end{bmatrix} \tag{9}$$

Where A is a matrix $n \times n$ whose entries are given by $A_{ij} = c_{M+i-j}$ with $a_k = 0$ if $k < 0$.

On the other hand, the coefficients $a_0, a_1, \cdots$ are determined from (6) and (8) with:

$$a_j = \sum_{k=0}^{j} c_{j-k} b_k\,, \quad 0 \le j \le M \tag{10}$$

Where $b_k = 0$ for $K > N$. Here, the resulting rational function $R_N^M(z)$ is called a Padé approximation.

2.2. PADÉ APPROXIMANT ALGORITHM

Semi-analytical methods for solving non-linear models require an initial approach to determine the solutions sought and the calculation of one or more fitting parameters. When the initial approach is chosen correctly, the results can be very precise, but not there is a general method for choosing such an initial approach [4],[5]. In this paper, we suggest using directly the serial solution of a non-linear model to find Padé's approximation with highly efficient results.

```
-----------------------------------------------------------------------
function [P,Q] = fraccion_continuaV2(M,N,y)
    J = M - N; t = 0;
    if(J<0)
        t = N; N = M; M = t; J = abs(J);
    end
    M = M + N - J; A = zeros(M+1); B = zeros(M+1); d = zeros(M+1,1);
    A(1,:) = y(J+1:end);
    B(1,1) = 1; d(1) = A(1,1); P0 = d(1); P1 = d(1);  Q0 = 1;
    for k = 1:M
        l = k-1;
        for i = 1:k
            A(i+1,l+1) = A(i,1)*B(i,l+2)-B(i,1)*A(i,l+2);
            B(i+1,l+1) = B(i,1)*A(i,l+1); l = l - 1;
        end
        d(k+1) = A(k+1,1)/B(k+1,1);
        if(k>1)
            P = suma_pol(P1,conv([d(k+1) 0],P0)); P0 = P1; P1 = P;
            Q = suma_pol(Q1,conv([d(k+1) 0],Q0)); Q0 = Q1; Q1 = Q;
        else
          Q1 = [d(2) 1];
        end
    end
    if(J>0)
        P = suma_pol(conv(fliplr(eval(y(1:J))),Q),conv([1 zeros(1,J)],P));
    end
    if(t~=0)
        t = P; P = Q; Q = t;
    end
end
-------------------------------------------------------------------------
```

This algorithm receives as input parameters, in addition to the degrees of numerator and denominator of the Padé approximation, the coefficients of the serial expansion of the non-linear model solution instead of the explicit function.

3. NUMERICAL RESULTS

The Riccati quadratic equation is a well-known asymptotic problem with some degree of difficulty in solving by other approach techniques [6].

3.1. NON-NORMAL PADÉ'S APPROACH

Let us assume the model

$$\begin{cases} u' - 2u + u^2 - 1 = 0 \\ u(0) = 0 \end{cases} \tag{1}$$

whose exact solution is given by

$$u(x) = 1 + \sqrt{2}\left(\sqrt{2}x + \frac{1}{2}\ln\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)$$

Now, let us assume initially that

$u(x) = \sum_{n=0}^{\infty} c_n x^n$,
then

$$\begin{aligned} u' - 2u + u^2 - 1 &= \sum_{n=0}^{\infty}(n+1)c_{n+1}x^n - 2\sum_{n=0}^{\infty} c_n x^n + \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} c_k c_{n-k}\right) x^n - 1 \\ &= \sum_{n=0}^{\infty}\left[(n+1)c_{n+1} - 2c_n + \sum_{k=0}^{n} c_k c_{n-k}\right] x^n - 1 \\ &= 0, \end{aligned}$$

From where, for $n = 0$, we obtain

$$c_1 - 2c_0 + c_0^2 = 1$$

And for $n > 0$ we have

$$(n+1)c_{n+1} - 2c_n + \sum_{k=0}^{n} c_k c_{n-k} = 0$$

Therefore, as $u(0) = 0$, then $c_0 = 0$.

However, if we consider the sequence $R_0^0, R_1^1, R_2^2, \ldots$ we can see that R_0^0 is not normal, in fact, $D_{0,0} = 1, D_{1,0} = 1, D_{0,1} = c_0,\ D_{1,1} = c_1$ and as $c_0 = 0$, then R_0^0 is not normal (See Figure1).

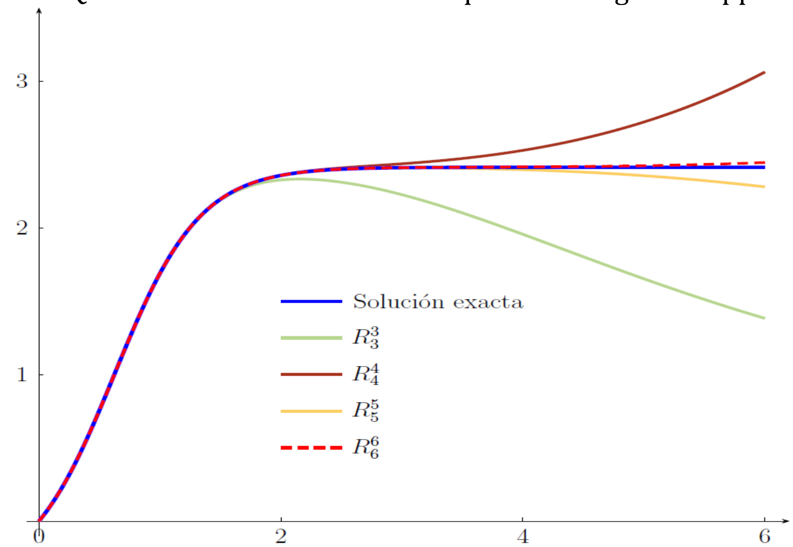

Figure 1: Exact solution and Padé's approximations of the model (1)

```
-----------------------------------------------------------------------------
clear all; global N
clc; N = 4; fun = @sistema1; y0 = zeros(1,2*N+3);
y = fsolve(fun,y0); [P,Q] = pade3(y,N+1,N+1);
x = 0:0.01:6; f = 1+sqrt(2)*tanh(sqrt(2).*x+1/2*log((sqrt(2)-1)/(sqrt(2)+1)));
y1 = polyval(P,x)./polyval(Q,x);
plot(x,f,x,y1)
-----------------------------------------------------------------------------
function F = sistema1(c)
global N
    F(1) = c(2) - 2*c(1) + c(1)*c(1) - 1;
    for n = 1:2*N+1
        F(n+1) = (n+1)*c(n+2) - 2*c(n+1);
        for k = 0:n
            F(n+1) = F(n+1) + c(k+1)*c(n-k+1);
        end
    end
    F(n+2) = c(1);
end
-----------------------------------------------------------------------------
function [P,Q] = pade3(y,M,N)
    A = zeros(N); w = zeros(N,1); P = zeros(M+1,1);
    for i = 1:N
        for j = 1:N
            if (M+1+i-j<=0)
                A(i,j) = 0;
            else
                A(i,j) = y(M+1+i-j);
            end
        end
        w(i) = -y(M+i+1);
    end
    Q = [1;A\w]; %q0=1;
    for j = 0:M
        for k = 0:j
            if k<=N
                P(j+1) = P(j+1) + y(j-k+1)*Q(k+1);
            else
                break;
            end
        end
    end
    for i = 1:floor((M+N+1)/2)
        temp = y(i); y(i) = y(M+N+2-i); y(M+N+2-i) = temp;
    end
    for i = 1:floor((M+1)/2)
        temp = P(i); P(i) = P(M+2-i); P(M+2-i) = temp;
    end
    for i = 1:floor((N+1)/2)
        temp = Q(i); Q(i) = Q(N+2-i); Q(N+2-i) = temp;
    end
end
-----------------------------------------------------------------------------
```

3.2. NORMAL PADÉ'S APPROACH

Let us assume the model

$$\begin{cases} u' - 2u + u^2 - 1 = 0 \\ u(0) = 1 \end{cases} \tag{2}$$

Whose exact solution is given by

$$u(x) = 1 + \sqrt{2}\tanh(\sqrt{2}x)$$

In this case, as

$$u(0) = 1,$$

Then $c_0 = 1$. Now, we can see that the sequence $R_0^0, R_1^0, R_1^1, \ldots$ is normal, we use this algorithm to determine the Padé's approximant (See Figure 2).

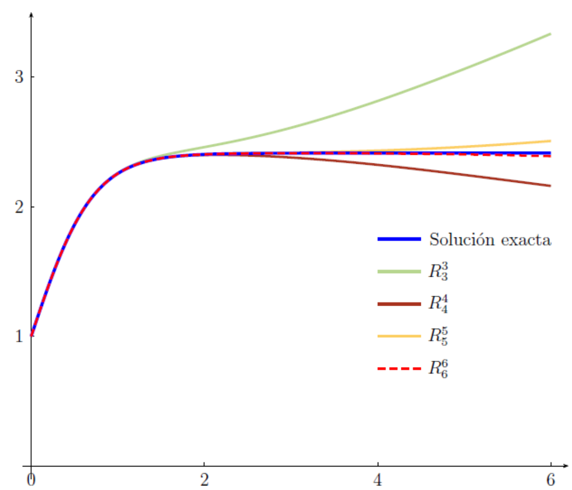

Figure 2: Exact solution and Padé's approximations of the model (2)

```
----------------------------------------------------------------------------
clear all; global N
clc; N = 6; fun = @sistema1; y0 = zeros(1,2*N+3); y = fsolve(fun,y0);
A = zeros(2*N+1); B = zeros(2*N+1); d = zeros(2*N+1,1);
A(1,:) = y(1:end-2); B(1,1) = 1; d(1) = A(1,1); P0 = d(1); P1 = d(1);  Q0 = 1;
x = 0:0.001:6; f = 1+sqrt(2)*tanh(sqrt(2).*x); plot(x,f,'b'); hold on;
for k = 1:2*N
    l = k-1;
    for i = 1:k
        A(i+1,l+1) = A(i,1)*B(i,l+2)-B(i,1)*A(i,l+2);
        B(i+1,l+1) = B(i,1)*A(i,l+1); l = l - 1;
    end
    d(k+1) = A(k+1,1)/B(k+1,1);
    if(k>1)
        P = suma_pol(P1,conv([d(k+1) 0],P0)); P0 = P1; P1 = P;
        Q = suma_pol(Q1,conv([d(k+1) 0],Q0)); Q0 = Q1; Q1 = Q;
    else
        Q1 = [d(2) 1];
    end
end
y1 = polyval(P,x)./polyval(Q,x);
plot(x,y1);
----------------------------------------------------------------------------
function F = sistema1(c)
global N
    F(1) = c(2) - 2*c(1) + c(1)*c(1) - 1;
    for n = 1:2*N+1
        F(n+1) = (n+1)*c(n+2) - 2*c(n+1);
        for k = 0:n
            F(n+1) = F(n+1) + c(k+1)*c(n-k+1);
        end
    end
    F(n+2) = c(1)-1;
end
----------------------------------------------------------------------------
```

4. CONCLUSION

In this work, we have illustrated the accuracy, simplicity and applicability of the Padé approach method used in different non-linear models. It was shown how Padé's approximations, calculated from the $M + N + 1$ terms of Taylor's series expansion of a function, improve the approximation of this one with respect to the one resulting from truncation of the same Taylor series.

Two algorithms were carried out to determine the coefficients of the rational expression corresponding to the approximate of Padé of order *(M,N)*: the first one by means of the solution of a system of linear equations for which the matrix of coefficients in a Toeplitz matrix very close to a singular matrix so solving the system may not be very efficient numerically; the second algorithm developed is much more efficient than the previous one because it uses the properties of continuous fractions to find Padé approximates of any order under the condition of normality of Padé's succession.

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None.

CONFLICT OF INTEREST

None.

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