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ON THE TERNARY QUADRATIC DIOPHANTINE EQUATION $6z^2 = 6x^2 - 5y^2$

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Abstract:

The ternary homogeneous quadratic equation given by $6z^2 = 6x^2 - 5y^2$ representing a cone is analyzed for its non-zero distinct integer solutions. A few interesting relations between the solutions and special polygonal and pyramided numbers are presented. Also, given a solution, formulas for generating a sequence of solutions based on the given solutions are presented.

Keywords:

Ternary quadratic, integer solutions, figurate numbers, homogeneous quadratic, polygonal number, pyramidal numbers.

Notations Used:

1. **Polygonal number of rank 'n' with sides m**

$$t_{m,n} = n\left(1 + \frac{(n-1)(m-2)}{2}\right) \qquad \textbf{m=3, 4, 5..., 10 ...}$$

2. **Stella octangular number of rank 'n'**

$$SO_n = n(2n^2 - 1)$$

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1. INTRODUCTION

The Diophantine equations offer an unlimited field for research due to their variety [1-3].In particular, one may refer [4-24] for quadratic equations with three unknowns.

This communication concerns with yet another interesting equation $6z^2 = 6x^2 - 5y^2$ representing non-homogeneous quadratic equation with three unknowns for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

2. METHOD OF ANALYSIS

The ternary quadratic Diophantine equation representing a cone under consideration is

$$6z^2 = 6x^2 - 5y^2 \tag{1}$$

To start with, it is seen that (1) is satisfied by the following non-zero integer triples (x, y, z):

(1921,2088,239);(466,504,74);(603,648,117);(439,468,101);

(341,360,91);(138,144,42); (169,168,71);(74,72,34);(87,72,57);(34,24,26).

As the considered equation is symmetric in x, y and z, we have presented only positive integer solutions for clear understanding.

However, we have other choices of solution to (1) which are illustrated below:

Choice 1: Introducing the linear transformations

$$\begin{aligned} x &= 6X + 5\alpha - 30\beta \\ y &= 6X + 6\alpha - 36\beta \\ z &= \alpha + 30\beta \end{aligned} \tag{2}$$

In (1), it is written as

$$X^2 = \alpha^2 + 180\beta^2 \tag{3}$$

Which is satisfied by

$$\begin{aligned} \beta &= 2pq \\ \alpha &= 180p^2 - q^2 \\ X &= 180p^2 + q^2 \end{aligned} \tag{4}$$

In view of (2), the non-zero distinct integer solutions to (1) are given by

$$\begin{aligned} x(p,q) &= 1980p^2 + q^2 - 60pq \\ y(p,q) &= 2160p^2 - 72pq \\ z(p,q) &= 180 - q^2 + 60pq \end{aligned} \tag{5}$$

Properties:

A few interesting properties obtained are as follows:

- $x(p, p+1) + y(p, p+1) - 4140t_{4,p} + 66(2p^2 - p) - 2t_{3,p} \equiv 1 (\mathrm{mod}\, 197)$
- $x(p+1, q) + z(p+1, q) - 4320t_{3,p} \equiv 1 (\mathrm{mod}\, 2160)$

- $x(p,1)+y(p,1)+z(p,1)-3960t_{4,p}-36t_{8,p}\equiv 0 \pmod{108}$
- $x(p,p)-1320t_{5,p}-t_{4,p}+15t_{10,p}\equiv 0 \pmod{705}$
- $y(p,p^2)-2160t_{4,p}+36SO_p\equiv 0 \pmod{36}$

Choice 2:

(3) is written as the system of double equations as given below

System 1 → $X+\alpha=\beta^2$, $X-\alpha=180$

System 2 → $X+\alpha=2\beta^2$, $X-\alpha=90$

System 3 → $X+\alpha=3\beta^2$, $X-\alpha=60$

System 4 → $X+\alpha=5\beta^2$, $X-\alpha=36$

System 5 → $X+\alpha=6\beta^2$, $X-\alpha=30$

System 6 → $X+\alpha=9\beta^2$, $X-\alpha=20$

System 7 → $X+\alpha=10\beta^2$, $X-\alpha=18$

System 8 → $X+\alpha=15\beta^2$, $X-\alpha=12$

System 9 → $X+\alpha=18\beta^2$, $X-\alpha=10$

Solving each of the above systems, the corresponding values of X, α and β are determined. Substituting these values of X, α and β in (2) the corresponding integer solutions to (1) are obtained which are exhibited in the table below.

Table: Solutions of (1)

SYSTEM	**x**	**y**	**z**
1	$22k^2$-60k+90	$24k^2$-72k	$2k^2$+60k-90
2	$11\beta^2$-30β+45	$12\beta^2$-36β	β^2+30β-45
3	$66k^2$-60k+30	$72k^2$-72k	$6k^2$+60k-30
4	$110k^2$-60k+18	$120k^2$-72k	$10k^2$+60k-18
5	$33\beta^2$-30β+15	$36\beta^2$-36β	β^2+30β-45
6	$198k^2$-60k+10	$216k^2$-72k	$18k^2$+60k-10
7	$55\beta^2$-30β+9	$60\beta^2$-36β	$5\beta^2$+30β-9
8	$330k^2$-60k+6	$360k^2$-72k	$30k^2$+60k-6
9	$99\beta^2$-30β+5	$108\beta^2$-36β	$9\beta^2$+30β-5

Choice 3:

In (3), it is written as

$$\alpha^2 + 180\beta^2 = X^2 = X^2 * 1 \tag{6}$$

Assume

$$X(a,b) = a^2 + 180b^2 \tag{7}$$

Write 1 as

$$1 = \frac{(2+i\sqrt{5})(2-i\sqrt{5})}{3^2} \tag{8}$$

Substituting (7) & (8) in (6) and applying the method of factorization define

$$(\alpha + i6\sqrt{5}\beta) = (a + i6\sqrt{5}b)^2 * \frac{(2+i\sqrt{5})}{3}$$

Equating the real and imaginary parts and replacing a by 6A and b by 6B, we have

$$\alpha = 24A^2 - 720AB - 4320B^2 \tag{9}$$

$$\beta = 2A^2 + 48AB - 360B^2 \tag{10}$$

and from (7), $X = 36\,A^2 + 6480\,B^2$

Substituting the above values of α, β, X in (2), the corresponding integer solutions to (1) are given by

$$x(A,B)=276A^2-5040AB+28080B^2$$
$$y(A,B)=288A^2-6048AB+25920B^2 \qquad (11)$$
$$z(A,B)=84A^2+720AB-15120B^2$$

Properties:

A few interesting properties obtained are as follows:

- $x(A+1,A)-y(A+1,A)-789t_{10,A}\equiv 996(\bmod 2343)$
- $x(A-1,A)-y(A-1,A)-z(A-1,A)-5824t_{8,A}\equiv 96(\bmod 11552)$
- $y(A+1,A)-z(A+1,A)-30660t_{4,A}+2544t_{7,A}-204=0$
- $z(A,A)=-14316\ t_{4,A}$
- $x(A,A)+z(A,A)-432t_{6,A}+77760t_{4,A}\equiv 0(\bmod 432)$

Note that instead of (8), one may also write 1 as

$$1=\frac{(-2+i\sqrt{5})(-2-i\sqrt{5})}{3^2}$$

For this choice, the corresponding integer solutions to (1) are given by

$$x(A,B)=36A^2-2160AB+71280B^2$$
$$y(A,B)=-2592AB+77760B^2$$
$$z(A,B)=36A^2-2160AB-6480B^2$$

3. REMARKABLE OBSERVATIONS

Let (x_0, y_0, z_0) be any given integer solution to (1). We illustrate below the method of obtaining a general formula for generating sequence of integer solutions based on the given solution.

Case (i):

Let
$$\begin{aligned} x_1 &= -2x_0 + h \\ y_1 &= 2y_0 + h \quad , \quad h \neq 0 \\ z_1 &= 2z_0 \end{aligned} \tag{12}$$

be the second solution of (1),Substituting (12) in (1) & performing a few calculations, we have

$$h = 20y_0 + 24x_0 \text{ and then}$$
$$x_1 = 22x_0 + 22y_0$$
$$y_1 = 22y_0 + 24x_0$$

This is written in the form of matrix as

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = M \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \tag{13}$$

where $M = \begin{pmatrix} 24 & 20 \\ 24 & 22 \end{pmatrix}$

Repeating the above process, the general solution (x_n, y_n) to (1) is given by

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = M^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \tag{14}$$

To find M^n, the eigen values of M are $\alpha = 1, \beta = 1$.
We know that

$$M^n = \frac{\alpha^n}{\alpha - \beta}(M - \beta I) + \frac{\beta^n}{\beta - \alpha}(M - \alpha I)$$

Using the above formula, we have

$$M^n = \frac{1}{4\sqrt{480}} \begin{bmatrix} 2\sqrt{480}(A^n) & 20(B^n) \\ 24(B^n) & 2\sqrt{480}(A^n) \end{bmatrix}$$

Thus the general solution (x_n, y_n, z_n) to (1) is given by

$$x_n = \frac{1}{4\sqrt{480}}\left[(2\sqrt{480}A^n)x_0 + 20B^n)y_0\right]$$

$$y_n = \frac{1}{4\sqrt{480}}\left[(24B^n)x_0 + (2\sqrt{480}A^n)y_0\right]$$

$$z_n = 2^n z_0$$

Where

$$A^n = (22+2\sqrt{480})^n + (22-2\sqrt{480})^n$$

$$B^n = (22+2\sqrt{480})^n - (22-2\sqrt{480})^n$$

Case (ii):

Let

$$x_1 = 3x_0 + h$$
$$y_1 = 3y_0, \qquad h \neq 0$$
$$z_1 = 3z_0 + h$$

Repeating the process as in the case (i) the corresponding general solution (x_n, y_n, z_n) to (1) is given by

$$x_n = \frac{1}{6}\left[(4(9)^n + 2(3)^n)x_0 + (2(9)^n + 2(3)^n)z_0\right]$$

$$y_n = 3^n y_0$$

$$z_n = \frac{1}{6}\left[(-4(9)^n - 4(3)^n)x_0 + (2(9)^n + 2(3)^n)z_0\right]$$

Case (iii):

Let

$$x_1 = x_0$$
$$y_1 = y_0 - 2h \qquad , h \neq 0$$
$$z_1 = z_0 + 2h$$

Repeating the procedure as presented in Case (i), the corresponding general solution (x_n, y_n, z_n) to (1) is given by

$$x_n = x_0$$

$$y_n = \frac{-1}{22}\left[(10(21)^n - 12(1)^n)y_0 + (2(-21)^n - 2(1)^n)z_0\right]$$

$$z_n = \frac{-1}{22}\left[(10(-21)^n - 10(1)^n)y_0 + (12(-21)^n + 10(1)^n)z_0\right]$$

4. CONCLUSION

In this paper, we have obtained infinitely many non-zero distinct integer solutions to the ternary quadratic Diophantine equation represented by $6z^2 = 6x^2 - 5y^2$

As quadratic equations are rich in variety, one may search for their choices of quadratic equation with variables greater than or equal to 3 and determine their properties through special numbers.

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