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# PROPERTIES OF SEQUENCES RELATED TO BALANCING NUMBERS

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## ABSTRACT

In this paper, we discuss a new approach to find the sequence of Balancing Numbers. We discuss some properties of two special sequences $\{u_n\}$ and $\{v_n\}$ related to the sequence of Balancing Numbers. W investigate the relations of these sequences with the sequence of Balancing Numbers. Also, we give some new identities of Balancing Numbers.

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## 1. INTRODUCTION

Balancing numbers is a natural number $B$ which satisfies the equation:$1 + 2 + 3 + \cdots + (B - 1) = (B + 1) + (B + 2) + \cdots + (B + k)$, where $k$ is called balancer. For example, $B = 6,35,204$ are balancing numbers and $k = 2,14,84$ respectively are balancers. A. Behera and G.K Panda [1] introduced the concept of balancing numbers in 1998.

The concept of balancing numbers is closely related with the established triangular numbers. The positive integer $B$ is called balancing number if and only if $B^2$ is a triangular number. Even if from the definition of balancing numbers, balancing numbers are positive integers greater than 1, 1 is accepting as balancing number. A. Behera and G.K Panda [1] proved that the sum of first odd $n$ balancing numbers is equal to the square of the $n^{th}$ balancing number, just like sum of first odd $n$ natural numbers is $n^2$.

The search for balancing numbers is well known integer sequence was first initiated by Liptai [2]. He proved that there is no balancing number in the Fibonacci sequence other than 1. He further showed that sequence balancing numbers in odd natural numbers are the sums of two consecutive balancing numbers.

## 2. BALANCING NUMBERS

The following theorem shows the new approach of finding the sequence of Balancing Numbers.

### 2.1. THEOREM

If $B_n$ is the $n^{th}$ Balancing Number, then $B_n = u_n v_n$, where $v_n + u_n = (1+\sqrt{2})^n$.

**Proof:** Let $B_n$ is the $n^{th}$ Balancing Number, then

$1 + 2 + 3 + \cdots + (B_n - 1) = (B_n + 1) + (B_n + 2) + (B_n + 3) + \cdots + (B_n + k_n)$, for some $k_n \in \mathbb{N}$

$$\therefore \sum_{i=1}^{B_n - 1} i = \sum_{i=1}^{B_n + k_n} i - \sum_{i=1}^{B_n} i$$

$$\therefore \frac{(B_n - 1)B_n}{2} = \frac{(B_n + k_n)(B_n + k_n + 1)}{2} - \frac{B_n(B_n + 1)}{2}$$

$$\therefore B_n^2 - B_n = B_n^2 + 2B_n K_n + k_n^2 + B_n + k_n - B_n^2 - B_n$$

$$\therefore k_n^2 + (2B_n + 1)k_n - B_n(B_n - 1) = 0 \cdots (1)$$

Equation (1) is a quadratic equation of $k_n$.

$$\therefore k_n = \frac{-(2B_n + 1) \pm \sqrt{8B_n^2 + 1}}{2}$$

But $k_n$ is a positive integer,

$$\therefore k_n = \frac{\sqrt{8B_n^2 + 1} - (2B_n + 1)}{2} \qquad \cdots (2)$$

As $k_n$ is a positive integer, $8B_n^2 + 1$ must be a perfect square of an odd integer.

Let $8B_n^2 + 1 = (4m_n \pm 1)^2$ for some $m_n \in \mathbb{N}$

$\therefore B_n^2 = m_n(2m_n \pm 1)$.

$\therefore m_n(2m_n \pm 1)$ is a perfect square, but $gcd(m_n, 2m_n \pm 1) = 1$

$\therefore$ Both $m_n$ and $2m_n \pm 1$ are perfect squares.

Let $m_n = u_n^2$ and $2m_n \pm 1 = v_n^2$

$$\therefore v_n^2 - 2u_n^2 = \pm 1 \qquad \cdots (3)$$

If $(v_n, u_n)$ is the $n^{th}$ solution of the equation (3), then

$$v_n + \sqrt{2}u_n = \left(1 + \sqrt{2}\right)^n, n \in \mathbb{N}$$

Now from equation (3), $B_n^2 = m_n(2m_n \pm 1) = u_n^2 v_n^2$

$$\therefore B_n = u_n v_n$$

$\therefore$ If $B_n$ is the $n^{th}$ Balancing Number, then

$B_n = u_n v_n$, where $v_n + \sqrt{2}u_n = \left(1 + \sqrt{2}\right)^n, n \in \mathbb{N}$ ■

Now, Let $C_n = B_n + k_n$

$$\therefore C_n = u_n v_n + \frac{\sqrt{8B_n^2 + 1} - (2B_n + 1)}{2}$$

$$= u_n v_n + \frac{(4m_n \pm 1) - (2B_n + 1)}{2}$$

$$= u_n v_n + \frac{4u_n^2 + (-1)^n - 2u_n v_n - 1}{2}$$

$$= \frac{4u_n^2 + (-1)^n - 1}{2}$$

$\therefore C_n = 2u_n^2$, if $n$ is even

$= 2u_n^2 - 1$, if $n$ is odd.

First Few Balancing Numbers are Shown in yhe Following Table.

**Table 1 First Seven Balancing Numbers**

| $n$ | $u_n$ | $v_n$ | $B_n = u_n v_n$ | $C_n$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 3 | 6 | 8 |
| 3 | 5 | 7 | 35 | 49 |
| 4 | 12 | 17 | 204 | 288 |
| 5 | 29 | 41 | 1189 | 1681 |
| 6 | 70 | 99 | 6930 | 9800 |
| 7 | 169 | 239 | 40391 | 57121 |

Now we know that $B_n = u_n v_n$, where $v_n + \sqrt{2}u_n = \left(1+\sqrt{2}\right)^n, n \in \mathbb{N}$

$B_{n+1} = u_{n+1}v_{n+1}$, where $v_{n+1} + \sqrt{2}u_{n+1} = \left(1+\sqrt{2}\right)^{n+1}, n \in \mathbb{N}$

$$\therefore v_{n+1} + \sqrt{2}u_{n+1} = \left(1+\sqrt{2}\right)^n\left(1+\sqrt{2}\right)$$
$$= \left(v_n + \sqrt{2}u_n\right)\left(1+\sqrt{2}\right)$$
$$= (2u_n + v_n) + \sqrt{2}(u_n + v_n)$$

$\therefore u_{n+1} = u_n + v_n$ and $v_{n+1} = 2u_n + v_n = u_n + u_{n+1}$ $\cdots (4)$

Also it is well-known that $B_1 = 1, B_2 = 6$ and $B_{n+1} = 6B_n - B_{n-1}$ for $n > 2$ [3].

## 3. PROPERTIES OF SEQUENCE $\{u_n\}$:

### 3.1. THEOREM

For the sequence $\{u_n\}$

1) $u_1 = 1, u_2 = 2$ and $u_n = 2u_{n-1} + u_{n-2}$ for $n > 2$
2) $u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$, where$\alpha$ and $\beta$ are the roots of the equation $x^2 - 2x - 1 = 0$.

**Proof:** (1)

We have $u_n = u_{n-1} + v_{n-1}$

$= (u_{n-2} + v_{n-2}) + (2u_{n-2} + v_{n-2})$ from result (4)

$= 2(u_{n-2} + v_{n-2}) + u_{n-2}$

$= 2u_{n-1} + u_{n-2}$

(2) We have $u_n = 2u_{n-1} + u_{n-2}$ $\therefore u_n - 2u_{n-1} - u_{n-2} = 0$

Let $u_n = Ax^n$, where $A$ is a constant.

$\therefore Ax^n - 2Ax^{n-1} - Ax^{n-2} = 0$ $\therefore x^2 - 2x - 1 = 0$

Let $\alpha$ and $\beta$ are the roots of the above equation.

$\therefore u_n = A\alpha^n + B\beta^n$, where $A$ and $B$ are a constants.

For $n = 1, u_1 = A\alpha + B\beta$ $\therefore A\alpha + B\beta = 1$

For $n = 2, u_2 = A\alpha^2 + B\beta^2$ $\therefore A\alpha^2 + B\beta^2 = 2$

Solving these two equations, we have $A = \frac{2-\beta}{\alpha(\alpha-\beta)}$ and $B = \frac{\alpha-2}{\beta(\alpha-\beta)}$

$$\therefore u_n = \frac{2-\beta}{\alpha(\alpha-\beta)}\alpha^n + \frac{\alpha-2}{\beta(\alpha-\beta)}\beta^n$$

$= \frac{\alpha\alpha^n}{\alpha(\alpha-\beta)} + \frac{-\beta\beta^n}{\beta(\alpha-\beta)} \quad (\because \alpha+\beta=2)$

$= \frac{\alpha^n-\beta^n}{\alpha-\beta}$ ■

### 3.2. THEOREM

For all $m, n \in \mathbb{N}$, (1) $gcd(u_n, u_{n+1}) = 1$, (2) $u_{m+n} = u_{m-1}u_n + u_m u_{n+1}$.

**Proof:** (1) Let $gcd(u_n, u_{n+1}) = d \quad \therefore d|u_n$ and $d|u_{n+1} d|(u_{n+1} - 2u_n) d|u_{n-1}$

similarly $d|u_n$ and $d|u_{n-1} d|u_{n-2}$

By Continuing this process, $d|u_1 d|1 d = 1$

(2) We will prove this result by induction on $n$.

For $n = 1$, $u_{m-1}u_1 + u_m u_2 = u_{m-1} + 2u_m = u_{m+1} \quad \therefore$ The result is true for $n = 1$.

Let the result is true for $n \leqslant k$

$\therefore u_{m+k-1} = u_{m-1}u_{k-1} + u_m u_k$ and $u_{m+k} = u_{m-1}u_k + u_m u_{k+1}$

Now for $n = k + 1$,

$u_{m+k+1} = 2u_{m+k} + u_{m+k-1}$

$= 2(u_{m-1}u_k + u_m u_{k+1}) + u_{m-1}u_{k-1} + u_m u_k$

$= u_{m-1}(2u_k + u_{k-1}) + u_m(2u_{k+1} + u_k)$

$= u_{m-1}u_{k+1} + u_m u_{k+2}$

Therefore the result is true for all $n$. ■

### 3.3. THEOREM

For the sequence $\{u_n\}$

1) $u_{mn}$ is divisible by $u_m$ for all $m, n \in \mathbb{N}$.
2) If $m = qn + r$, then $gcd(u_m, u_n) = gcd(u_r, u_n)$ for all $m, n \in \mathbb{N}, m \geqslant n$.
3) $gcd(u_m, u_n) = u_d$ where $d = gcd(m, n)$ for all $m, n \in \mathbb{N}$.

**Proof:** (1) We will prove this result by induction on $n$.

For $n = 1$ the result is true. Let the result is true for $n \leqslant k$.

Now for $n = k + 1$, $u_{m(k+1)} = u_{mk+m} = u_{mk-1}u_m + u_{mk}u_{m+1}$

But $u_m|u_{mk} \quad \therefore u_m|u_{m(k+1)}$

(2) $gcd(u_m, u_n) = gcd(u_{qn+r}, u_n) = gcd(u_{qn-1}u_r + u_{qn}u_{r+1}, u_n)$

$= gcd(u_{qn-1}u_r, u_n) \quad (\because u_n|u_{qn})$

$= gcd(u_r, u_n) \quad (\because u_n|u_{qn}$ and $gcd(u_{qn}, u_{qn-1}) = 1)$

(3) Let $m \geqslant n$. Applying the Euclidean algorithm to $m$ and $n$, we have

$m = q_1 n + r_1 \quad 0 < r_1 < n$

$n = q_2 r_1 + r_2 \quad 0 < r_2 < r_1$

$r_1 = q_3 r_2 + r_3 \quad 0 < r_3 < r_2$

….. ….. …..

$r_{n-2} = q_n r_{n-1} + r_n \quad 0 < r_n < r_{n-1}$

$r_{n-1} = q_{n+1} r_n + 0$

$\therefore gcd(u_m, u_n) = gcd(u_{r_1}, u_n) = gcd(u_{r_1}, u_{r_2}) = \cdots = gcd(u_{r_{n-1}}, u_{r_n})$

But $r_n | r_{n-1} u_{r_n} | u_{r_{n-1}} gcd(u_{r_{n-1}}, u_{r_n}) = u_{r_n} = u_d$ as $r_n = gcd(m, n) = d$

$\therefore gcd(u_m, u_n) = u_d$ ■

## 3.4. THEOREM

For the sequence $\{u_n\}$

1) $u_n^2 - u_{n+1}u_{n-1} = (-1)^{n-1}$.
2) $u_{n+1}u_{n+2} - u_n u_{n+3} = 2(-1)^n$.
3) $u_n u_{n+1} u_{n+2} u_{n+3} + 1$ is always a perfect square.
4) $u_{2n} = 2B_n$.
5) $u_{2n+1} = B_{n+1} - B_n$.

**Proof:** (1)

$$\begin{aligned} u_n^2 - u_{n+1}u_{n-1} &= u_n(2u_{n-1} + u_{n-2}) - u_{n+1}u_{n-1} \\ &= -(u_{n+1} - 2u_n)u_{n-1} + u_n u_{n-2} \\ &= -u_{n-1}^2 + u_n u_{n-2} \\ &= (-1)(u_{n-1}^2 - u_n u_{n-2}) \end{aligned}$$

Continuing this process, after $n-2$ steps we get

$$\begin{aligned} u_n^2 - u_{n+1}u_{n-1} &= (-1)^{n-2}(u_2^2 - u_3 u_1) \\ &= (-1)^{n-2}(4-5) \\ &= (-1)^{n-1} \end{aligned}$$

(2)

$$\begin{aligned} u_{n+1}u_{n+2} - u_n u_{n+3} &= u_{n+1}u_{n+2} - u_n(2u_{n+2} + u_{n+1}) \\ &= u_{n+2}(u_{n+1} - 2u_n) - u_n u_{n+1} \\ &= u_{n+2}u_{n-1} - u_n u_{n+1} \\ &= (2u_{n+1} + u_n)u_{n-1} - u_n(2u_n + u_{n-1}) \\ &= 2u_{n+1}u_{n-1} + u_n u_{n-1} - 2u_n^2 - u_n u_{n-1} \\ &= -2(2u_n^2 - u_{n+1}u_{n-1}) \\ &= -2(-1)^{n-1} \\ &= 2(-1)^n \end{aligned}$$

(3)

$$\begin{aligned} u_n u_{n+1} u_{n+2} u_{n+3} + 1 &= (u_n u_{n+3})(u_{n+1}u_{n+2}) + 1 \\ &= (u_{n+1}u_{n+2} - 2(-1)^n)(u_{n+1}u_{n+2}) + 1 \\ &= (u_{n+1}u_{n+2})^2 - 2(-1)^n u_{n+1}u_{n+2} + 1 \\ &= (u_{n+1}u_{n+2} - (-1)^n)^2 \end{aligned}$$

(4) By substituting $m = n$ in the result $u_{m+n} = u_{m-1}un + u_m u_{n+1}$, we have

$$\begin{aligned} u_{n+n} &= u_{n-1}u_n + u_n u_{n+1} \\ u_{2n} &= u_n(u_{n-1} + u_{n+1}) \\ &= u_n(u_{n-1} + u_n + v_n) \\ &= u_n(v_n + v_n) \\ &= 2u_n v_n \\ &= 2B_n \end{aligned}$$

(5) The result is true for $n = 1$. Let the result is true for $n \leqslant k$.

$\therefore u_{2k+1} = B_{k+1} - B_k$.

Now for $n = k + 1$, we have

$$u_{2k+3} = u_{(2k+1)+2}$$
$$= u_{2k}u_2 + u_{2k+1}u_3$$
$$= (2B_k)2 + (B_{k+1} - B_k)5$$
$$= 5B_{k+1} - B_k$$
$$= 6B_{k+1} - B_k - B_{k+1}$$
$$= B_{k+2} - B_{k+1}$$

Therefore the result is true for all $n$. ■

## 3.5. THEOREM

1) $u_1 + u_3 + u_5 + \cdots + u_{2n-1} = B_n$
2) $u_2 + u_4 + u_6 + \cdots + u_{2n} = \frac{u_{2n+1}-1}{2}$
3) $u_1 + u_2 + u_3 + \cdots + u_n = \frac{u_{n+1}+u_n-1}{2}$
4) $u_n^2 + u_{n+1}^2 = u_{2n+1}$
5) $u_1^2 + u_2^2 + u_3^2 + \cdots + u_n^2 = \frac{u_n u_{n+1}}{2}$

**Proof:** (1) We know that $u_{2n+1} = B_{n+1} - B_n$.

$\therefore\ u_{2n-1} = B_n - B_{n-1}$

$u_{2n-3} = B_{n-1} - B_{n-2}$

$u_{2n-5} = B_{n-2} - B_{n-3}$

….. ….. ….. …..

$u_5 = B_3 - B_2$

$u_3 = B_2 - B_1$

Adding these results, we have,

$u_3 + u_5 + \cdots + u_{2n-1} = B_n - B_1$

$\therefore u_1 + u_3 + u_5 + \cdots + u_{2n-1} = B_n \quad (\because u_1 = B_1 = 1)$

(2) We know that $u_{n+1} = 2u_n + u_{n-1}$ for $n \geqslant 2$. $\therefore 2u_n = u_{n+1} - u_{n-1}$.

By substituting $2n$ in place of $n$, we have

$2u_{2n} = u_{2n+1} - u_{2n-1}$

$2u_{2n-2} = u_{2n-1} - u_{2n-3}$

$2u_{2n-4} = u_{2n-3} - u_{2n-5}$

….. ….. ….. …..

$2u_4 = u_5 - u_3$

$2u_2 = u_3 - u_1$

Adding these results, we have,

$2u_2 + 2u_4 + 2u_6 + \cdots + 2u_{2n} = u_{2n+1} - u_1$

$\therefore u_2 + u_4 + u_6 + \cdots + u_{2n} = \frac{u_{2n+1}-1}{2}$

(3) We know that $u_{n+1} = 2u_n + u_{n-1}$ for $n \geqslant 2$.

$\therefore 2u_n = u_{n+1} - u_{n-1}$

$2u_{n-1} = u_n - u_{n-2}$

$2u_{n-2} = u_{n-1} - u_{n-3}$

….. ….. ….. …..

$2u_4 = u_5 - u_3$

$2u_3 = u_4 - u_2$

$2u_2 = u_3 - u_1$

Adding these results, we have,

$2u_2 + 2u_3 + 2u_4 + \cdots + 2u_n = u_{n+1} + u_n - u_2 - u_1$

$2u_1 + 2u_2 + 2u_3 + 2u_4 + \cdots + 2u_n = u_{n+1} + u_n - u_2 - u_1 + 2u_1$

$\therefore u_1 + u_2 + u_3 + \cdots + u_n = \frac{u_{n+1}+u_n-1}{2}$ $(\because u_1 = 1, u_2 = 2)$

(4) We know that $u_{m+n} = u_{m-1}u_n + u_m u_{n+1}$.

By substituting $m = n + 1$ in the above result, we have

$u_{n+1+n} = u_n u_n + u_{n+1}u_{n+1}$. $\therefore u_{2n+1} = u_n^2 + u_{n+1}^2$.

(5) From the above result, we have,

$u_n^2 + u_{n-1}^2 = u_{2n-1}$

$u_{n-1}^2 + u_{n-2}^2 = u_{2n-3}$

$u_{n-2}^2 + u_{n-3}^2 = u_{2n-5}$

….. ….. ….. …..

$u_3^2 + u_2^2 = u_5$

$u_2^2 + u_1^2 = u_3$

Adding these results, we have,

$u_1^2 + 2(u_2^2 + u_3^2 + \cdots + u_{n-1}^2) + u_n^2 = u_3 + u_5 + u_7 + \cdots + u_{2n-1}$

$\therefore 2(u_1^2 + u_2^2 + u_3^2 + \cdots + u_n^2) = u_1 + u_3 + u_5 + u_7 + \cdots + u_{2n-1} + u_n^2$

$= \frac{u_{2n}}{2} + u_n^2$

$= \frac{u_{2n}+2u_n^2}{2}$

$= \frac{u_{n-1}u_n+u_n u_{n+1}+2u_n^2}{2}$

$= \frac{u_n u_{n+1}+u_n(2u_n+u_{n-1})}{2}$

$= \frac{u_n u_{n+1}+u_n u_{n+1}}{2}$

$= u_n u_{n+1}$

$\therefore u_1^2 + u_2^2 + u_3^2 + \cdots + u_n^2 = \frac{u_n u_{n+1}}{2}$ ■

## 4. PROPERTIES OF SEQUENCE $\{v_n\}$

### 4.1. THEOREM

For the sequence $\{v_n\}$

1) $v_1 = 1, v_2 = 3$ and $v_n = 2v_{n-1} + v_{n-2}$ for $n > 2$
2) $v_n = \frac{\alpha^n+\beta^n}{\alpha+\beta}$, where$\alpha$ and $\beta$ are the roots of the equation $x^2 - 2x - 1 = 0$.

**Proof:** (1)

We have $v_n = 2u_{n-1} + v_{n-1}$

$= 2(u_{n-2} + v_{n-2}) + v_{n-1}$

$= 2u_{n-2} + 2v_{n-2} + v_{n-1}$

$= 2u_{n-2} + v_{n-2} + v_{n-2} + v_{n-1}$
$= v_{n-1} + v_{n-2} + v_{n-1}$
$= 2v_{n-1} + v_{n-2}$

(2) The proof of this property is similar to the proof of the corresponding property of sequence $\{u_n\}$.

■

## 4.2. THEOREM

For the sequence $\{v_n\}$

1) $gcd(v_n, v_{n+1}) = 1$
2) $v_n^2 - v_{n-1}v_{n+1} = 2(-1)^n$.
3) $v_{n+1}v_{n+2} - v_n v_{n+3} = 4(-1)^n$.
4) $v_n v_{n+1} v_{n+2} v_{n+3} + 4$ is always a perfect square.

**Proof:** The proofs of all these properties are similar to the proofs of the corresponding properties of sequence $\{u_n\}$.
■

## 4.3. THEOREM

For the sequence $\{v_n\}$

1) $2v_{2n} = B_{n+1} - B_{n-1}$.
2) $v_{2n+1} = B_{n+1} + B_n$.

**Proof**

(1) Wehave $v_{2n} = u_{2n-1} + u_{2n}$

$\therefore 2v_{2n} = 2u_{2n-1} + 2u_{2n}$
$= 2(B_n - B_{n-1}) + 4B_n$
$= 6B_n - 2B_{n-1}$
$= 6B_n - B_{n-1} - B_{n-1}$
$= B_{n+1} - B_{n-1}$

(2) Wehave $v_{2n+1} = u_{2n} + u_{2n+1}$
$= 2B_n + B_{n+1} - B_n$
$= B_{n+1} + B_n$

■

## 4.4. THEOREM

1) $v_1 + v_3 + v_5 + \cdots + v_{2n-1} = \frac{B_{n+1} - B_{n-1} - 2}{4}$
2) $v_2 + v_4 + v_6 + \cdots + v_{2n} = \frac{B_{n+1} + B_n - 1}{2}$
3) $v_1 + v_2 + v_3 + \cdots + v_n = \frac{v_{n+1} + v_n - 1}{2}$

**Proof:** (1) We know that $v_{2n} = 2v_{2n-1} + v_{2n-2}$.

$\therefore 2v_{2n-1} = v_{2n} - v_{2n-2}$
$2v_{2n-3} = v_{2n-2} - v_{2n-4}$
$2v_{2n-5} = v_{2n-4} - v_{2n-6}$
….. ….. ….. …..
$2v_5 = v_6 - v_4$

$2v_3 = v_4 - v_2$

Adding these results, we have,

$2(v_3 + v_5 + v_7 + \cdots + v_{2n-1}) = v_{2n} - v_2$

$\therefore 2(v_1 + v_3 + v_5 + v_7 + \cdots + v_{2n-1}) = v_{2n} - 1$

$\therefore v_1 + v_3 + v_5 + v_7 + \cdots + v_{2n-1} = \frac{v_{2n}-1}{2} = \frac{2v_{2n}-2}{4} = \frac{B_{n+1}-B_{n-1}-2}{4}$

(2)

Weknowthat $2v_{2n} = B_{n+1} - B_{n-1}$

$\therefore \sum_{i=2}^{n} 2v_{2i} = \sum_{i=2}^{n} (B_{i+1} - B_{i-1})$

$= (B_3 - B_1) + (B_4 - B_2) + (B_5 - B_3) + \cdots + (B_n - B_{n-2}) + (B_{n+1} - B_{n-1})$

$= B_{n+1} + B_n - B_2 - B_1$

$= B_{n+1} + B_n - 4$

$\therefore \sum_{i=1}^{n} 2v_{2i} = B_{n+1} + B_n - 7 + 2v_2$

$= B_{n+1} + B_n - 1$

$\therefore \sum_{i=1}^{n} v_{2i} = \frac{B_{n+1}+B_n-1}{2}$

(3) The proof of this property is similar to the proof of the corresponding property of sequence $\{u_n\}$. ■

## 5. CONCLUDING REMARKS

In this paper, we have discussed a new approach of finding the sequence of Balancing Numbers. We have found some properties of two special sequences $\{u_n\}$ and $\{v_n\}$ related to the sequence of Balancing Numbers. Also we have found some relations of the sequences $\{u_n\}$ and $\{v_n\}$ with the sequence of Balancing Numbers. Such more relations can also be found and it is an open area of research.

## CONFLICT OF INTERESTS

None.

## ACKNOWLEDGMENTS

None.

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