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INCORPORATING FUZZY GRAPHIC MATROID IN NETWORK CONSTRUCTION USING INDEX CODES

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ABSTRACT

A fuzzy graphic matroid can be represented over a finite field F. The necessary and sufficient conditions of a fuzzy graphic matroid to be binary are given. Also, discussed the fuzzy graph-theoretic context in which fuzzy matroids arise. Fuzzy matroid operations, which will illuminate more analogies that corresponds to the operations in fuzzy graph theory and matrix theory. The traditional network models stores the received packets and forward them without applying any additional process to the packets. The uncertainties caused during the transmission of messages in the traditional network models are rectified and noiseless transmission of messages in the network model is improved. The primary emphasis of this paper lies in the exploration of network codes and their correlation with fuzzy graphic matroid.

Keywords: Fuzzy Matroid, Graphic Matroid, Fuzzy Graphic Matroid, Uniform Fuzzy Graphic Matroid, Index Codes
AMS Classification: 05C40, 05C62, 05C72, 05C70, 05C90, 94C15.

1. INTRODUCTION

A graph is an abstraction of relationships that emerge in nature which is a collection of vertices connected by edges. Graphs serve as highly useful tool with a wide range of real-life applications. The basic ideas in graph theory were introduced in 18th century by Leonhard Euler. In 1847, Kirchoff developed the tree theory in networks [19]. Crisp graphs alone are inadequate for encapsulating the uncertainties associated with parameters in network modelling [7]. Hence, fuzzy graph theory constitutes a crucial research domain that forms the foundation for representing networks characterized by ambiguity. In 1965, Zadeh introduced the concept of fuzzy sets in his paper [21]. In 1973, Kaufmann was the pioneer in defining fuzzy graphs [6]. Then, in 1975, Rosenfeld expanded upon the theory of fuzzy graphs by incorporating fuzzy relations within fuzzy sets [7]. Mordeson and Nair defined arc disjoint fuzzy graphs and studied some of their properties.

The foundation of matroid theory primarily emerged from the principles of independence and vector spaces. Hassler Whitney introduced matroid theory in 1935 [20]. Whitney estab- lished two independence axioms and coined the term matroids to describe structures adhering to these axioms [16]. His insightful observation revealed that these axioms provided a unified abstraction of independence shared by both graphs and matrices [16]. Consequently, many of the terminologies in matroid theory closely resemble the concepts of linear algebra and graph theory. Shortly after Whitney's work, Saunders Mac Lane (1936) authored a significant arti- cle highlighting the relationship between matroids and projective geometry. In 1937, Van Der Waerden in his renowned textbook on Modern Algebra identified parallels between algebraic and linear dependence. Richard Rado further extended this theory in 1940, introducing the concept of independence systems. During the 1950s, Tutte emerged as the leading figure in the field of matroid theory, and his contributions to this discipline were abundant [14]. The theory of graphic matroids introduced by Whitney and further studied by Tutte in 1954 and others [16]. Graphic matroids are matroids that arise from graphs, capturing the combinatorial properties of graphs such as connectivity and planarity [16].

In 1988, Goetschal and Voxman introduced the notion of fuzzy matroids. Fuzzy matroids deals with graphical and algebraical structures related to the membership grades of a fuzzy graph [5]. A new class of matroids from fuzzy graphs was constructed by Sabana and Sameena in 2019 [12]. From a fuzzy graph G, subgraphs can be formed by deleting or contracting edges. The concept of representable fuzzy matroid and the idea of inducing fuzzy matroids from fuzzy vector space was done by Sabana and Sameena in 2021 [14]. The isomorphism between two fuzzy matroids and the properties are defined by Sabana and Sameena in 2021[13].

A matroid is representable over some finite field. Anushka Murthy characterised binary matroids and regular matroids [9]. In this paper, the representation of fuzzy matroids over a finite field as a generalization of linear dependence is given. The characterization for binary fuzzy matroid and their necessary and sufficient conditions for a fuzzy matroid to be binary are illustrated. The issue of information transmission within a communication network featuring noiseless links, where the interdependence among data across various network edges adheres to fuzzy matroidal networks, is the subject of concern. The paper is organised as follows:

In Section 2, the prerequisite is given. Section 3 deals with the binary representation of a fuzzy matroid over finite field. The construction of network using fuzzy graphic matroids are explained briefly in Section 4. The network constructed using the index codes retrieving the messages from the receivers are discussed and the results are furnished in Section 5. Section 6 concludes the paper.

2. PRELIMINARIES

Definition 2.1. [10] Let I be a non-empty family of a finite subsets of E and the pair M = (E, I) is called a matroid if it satisfies the following properties:

1) $\varphi \in I$
2) If $A \in I$, $B \subset A$, then $B \in I$
3) If $A, B \in I$; $|B| \geq |A|$, then there exists an element $e \in B \setminus A$ such that $A \cup \{e\} \in I$.

A matroid is a combinatorial structure that provides a broader conceptual framework en- compassing the idea of linear independence. It serves as a unifying bridge between graph theory and linear algebra [18]. A matroid that is representable over every field is called regu- lar. A matroid is binary [9] if it can be represented over a finite field F2 and ternary if it can be represented over a finite field F3.

The collection of basis uniquely determines the matroid. A basis (or base) of a matroid refers to the largest independent set within M, while a circuit of the matroid corresponds to the smallest dependent set in M. The size of the largest independent set within the set E is defined to be the rank of a matroid and is represented as r(I).[10]

Definition 2.2. [10] If two matroids are defined to be isomorphic, it implies that the existence of a bijective mapping $\phi: E1 \rightarrow E2$ such that $\phi(C)$ is a circuit of M_2 if and only if a subset C of E1 is a circuit of M_1.

A graphic matroid is derived from graph G. It is defined on the egde set of the graph G and it is denoted by M (G).[18]

1) A basis or a base of M (G) is the spanning trees of G.
2) A circuit of M (G) is the simple cycles in a graph G.

The rank function of a graphic matroid M (G) defined on a graph G = (V, E) is given by r(M(G)) = n - K(V, F) where n is the number of verticec and K(V, F) is the number of con- nected components.[11]

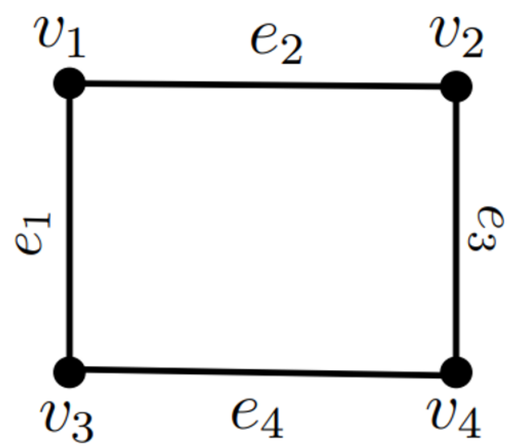

Figure 2.1 A Graph

Example 2.3. Let E = {e_1, e_2, e_3, e_4} and I = {φ, {e_1}, {e_2}, {e_3}, {e_4}{e_1, e_2}, {e_2, e_3}, {e_1, e_4}, {e_3, e_4} be the subsets of E. The bases of the graphic matroid in Figure 2.1 are B = {e_1, e_2, e_3}; {e_2, e_3, e_4}; {e_3, e_4, e_1} while the circuits of the graphic matroid in Figure 2.1is C = {e_1, e_2, e_3, e_4} and r(M (G)) = 3.

Definition 2.4. [10] A matrix A over a specified field F such that M ∼= M [A] then M is defined to be F-representable. This matrix A is referred to as an F -representation of M.

Example 2.5. The F -representation of the matroid M = (E, I) in terms of the independent edges of Figure 2.1 is given below:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

F - **representation**

A fuzzy graphic matroid is derived from a fuzzy graph. To understand this, we need to explore the definition of a fuzzy graph and fuzzy matroid.

Let G = (σ∗, µ∗) is a pair of functions with σ∗ and µ∗ where σ∗ : V −→ [0, 1] is a fuzzy vertex set of G and µ∗ : V × V −→ [0, 1] is a fuzzy edge set of G where µij ≤ min(µi, µj) for every µi, µj ∈ V . Then G = (σ∗, µ∗) is called as a fuzzy graph with n vertices and m edges.[8]

Let I be a fuzzy subset on E is a function I : X −→ [0, 1]. The family of fuzzy set on E is given by F (X). If P, Q ∈ F(E), then

1) supp(P) = {x ∈ E | P (x) > 0}, a crisp set

2) m(P) = min{P (x) | x ∈ supp(P)}
3) P ∪ Q = max{P (x), Q(x)}, x ∈ E
4) P ∩ Q = min{P (x), Q(x)}, x ∈ E.

Note: For the sake of convenience, different notations are used here.

Definition 2.6. [1, 5] Let E be a finite set and let I ∈ F(E) satisfying the following conditions:
1) φ ∈ I
2) P ⊂ Q and Q ∈ P, where P ⊂ Q, P (x) ≤ Q(x) forall x ∈ E
3) If P, Q ∈ I with |supp (P)| ≤ |supp (Q)|, there exists R ∈ I such that
- P ⊂ R ⊆ P ∪ Q
- m(R) = min {m(P), m(Q)}.

Then FM = (E, I) is a fuzzy matroid.

Definition 2.7. [2] A fuzzy graphic matroid is derived from a fuzzy graph G(σ, μ). It is defined on the edge set of the fuzzy graph G(σ, μ) and it is denoted by G(σ, μ, M) such that
1) A basis or base of M(G) is the spanning trees of a fuzzy graph G(σ, μ).
2) A circuit of M(G) is the simple cycles in a fuzzy graph G(σ, μ).

Definition 2.8. [2] The rank of a fuzzy graphic matroid is the maximum sum of the edges in an independent set and it is denoted by R.

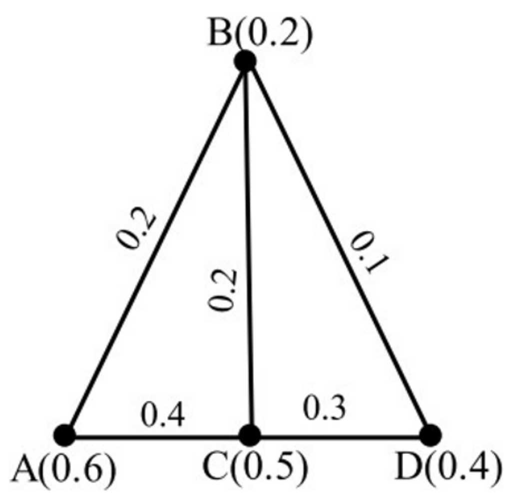

Figure 2.2 A Fuzzy Graph G

Example 2.9. In Figure 2.2, E = {(μ_{12}, 0.2), (μ23, 0.2), (μ_{24}, 0.1), (μ_{13}, 0.4), (μ_{34}, 0.3)} be a finite set and P = {(μ_{12}, 0.2), (μ_{13}, 0.4)}, Q = {(μ_{12}, 0.2), (μ_{13}, 0.4), (μ_{34}, 0.3)} ∈ I is the independent fuzzy subsets of E where P ⊂ Q and |supp(P)| = 2,

$|supp(Q)| = 3$ such that $|supp(P)| < |supp(Q)|$ there exists

$R = \{(\mu_{23}, 0.2), (\mu_{13}, 0.4), (\mu_{34}, 0.3)\}$ and $P \subset R \subseteq (P \cup Q)$ where $(P \cup Q) = \{(\mu_{12}, 0.2), (\mu_{13}, 0.4)$

$, (\mu 34, 0.3)\}$ and $m(R) = \min\{m(P)x, m(Q)x\} = 0.2$.

The bases of the fuzzy graphic matroid in Figure 2.2 are $Q = \{\{(\mu_{12}, 0.2), (\mu_{13}, 0.2), (\mu_{24}, 0.3)\},$

$\{(\mu_{12}, 0.2), (\mu_{24}, 0.2), (\mu_{34}, 0.2)\}\{(\mu_{12}, 0.2), (\mu_{13}, 0.4), (\mu_{24}, 0.3)\}\}$ while the circuits of the fuzzy graphic matroid in Figure 2.2 are $C = \{\{(\mu_{12}, 0.2), (\mu_{13}, 0.1), (\mu_{23}, 0.2)\},$

$\{(\mu_{23}, 0.2), (\mu_{24}, 0.1), (\mu_{34}, 0.3)\}\}$.

The dual of fuzzy graphic matroid $G(\sigma, \mu, M)$ is $G(\sigma, \mu, M)*$ whose bases are the complements of the bases $G(\sigma, \mu, M)$.

- The sub-bases S of $G(\sigma, \mu, M)$ are the fuzzy set of edges containing a base.
- The hypobase H of $G(\sigma, \mu, M)$ are the maximal fuzzy set of edges containing no base.
- $Q*$, $C*$ are the cobases and cocircuits of Fuzzy Graphic Matroid.

A minor of a fuzzy graphic matroid $G(\sigma, \mu, M)$ is formed by two operations:

- The restriction that deletes an edge μij of a fuzzy graph without changing the indepen- dence or rank of the remaining subset of the edge set of G.
- The contraction that deletes an edge μij of a fuzzy graph by decreasing the rank of every edge set it belongs to.

The span of a fuzzy graphic matroid consists of all the edges that can be added to a given set of edges without forming a cycle in the fuzzy graph G.

$$\sigma(G(\sigma, \mu, M)) = \{\mu ij \in E | P \cup \{\mu ij\} \in I\}$$

A $G(\sigma, \mu, M)$ is said to be uniform fuzzy graphic matroid if all of its base (or basis) have the same number of edges which corresponds to the independent set of edges of a fuzzy graph G. It is denoted by Ur,n where r represents the number of edges in the basis and n represents the number of edges of a G.

3. BINARY REPRESENTATION OF FUZZY GRAPHIC MATROIDS

The binary representation of a fuzzy graphic matroid $G(\sigma, \mu, M)$ is a matrix D with entries over a finite field F such that there is an one-to-one correspondence between the columns of D and the ground set of edges of $G(\sigma, \mu, M)$, and a set of columns in D is linearly independent if and only if the corresponding edge set is independent in $G(\sigma, \mu, M)$.

Lemma 3.1. Let E be any finite fuzzy set of −→ over a finite field F and I be all collection of linearly independent vectors in E, then $G(\sigma, \mu, M) = (E, I)$ is a fuzzy graphic matroid.

Proof. The properties of FM states that

1) $\varphi \in I$
2) $P \subset Q$ and $Q \in P$, where $P \subset Q$, $P(x) \leq Q(x)$ forall $x \in E$

(i) and (ii) is obvious from the definition of fuzzy matroid. Let P and Q are independent sets belongs to I. The dim(P) and dim(Q) represents the number of elements in the set P and Q. If P is a subset of Q then $\dim(Q) > \dim(P)$. Let σ be the span function of $(P \cup Q)$, then $\dim(P) \leq \dim(\sigma)$. $Q \cup \{\mu ij\}$ is dependent for $\mu ij \in P$. Therfore, $\dim(P) \leq \dim(\sigma) \leq \dim(Q)$ which contradicts $\dim(Q) > \dim(P)$.

Let us prove the uniform fuzzy graphic matroid is F- representable over a field with $n \geq 2$.

Theorem 3.2. The uniform fuzzy graphic matroid U2,n is F- representable if and only if $|F| \geq n - 1$ over a field with $n \geq 2$.

Proof. Let [Ir|D] be a F representation of $U2,_n$. The entries of the first row of D are equal to one and the entries of second row of D are mutually distinct non zero elements of F and thus n − 2 ≤ |F | − 1. If D is a matrix with all entries in the first row equal to one and the entries of second row are mutually distinct non zero elements of F, then [Ir|D] is a F representation of D.

Corollary. If p is a prime number, then the fuzzy graphic matroids $U_{2,p}+_2$ and $U_{p,p+2}$ are excluded minors of F(2) representablity.

The next theorem addresses the isomorphic condition for a fuzzy graphic matroid to be binary.

Theorem 3.3. G(σ, μ, M) is binary if and only if U2,4 is not its minor.

Proof. Let G(σ, μ, M) and (σ, μ, M)∗ be the minor minimal non binary fuzzy graphic matroids. Let B be the base of G(σ, μ, M) and D be the B- fundamental circuit incidence matrix of G(σ, μ, M).

If G(σ, μ, M) is binary, [Ir|D] is its vector representation whereas G(σ, μ, M) is not binary then [I |D] differs from G(σ, μ, M). B and B/ differs in one edge B/ = (B − {μ }) + {μ }. The initial entry in D is non-zero. The edges set E and the subset {μjk} corresponds to the first column of Ir and D. By adding the first row of [Ir|D] to every row containing a non-zero entry in the first column of D, replace the columns of Ir and D with [Ir|Db] representing the fuzzy graphic matroid. D differs from the partial representation concerning B/ . Consequently,

G(σ, μ, M) is a minor-minimal fuzzy graphic matroid that is not binary. Therefore, G(σ, μ, M)

must be isomorphic to $U_{2,4}$.

The following theorem describes the symmetric difference of any two circuits within a fuzzy graphic matroid.

Theorem 3.4. Let G(σ, μ, M) be a fuzzy graphic binary matroid, then the symmetric difference of any two circuits within a fuzzy graphic binary matroid results in a set that is a disjoint union of circuits of G(σ, μ, M).

Proof. Let Ci and Cj be the circuits of G(σ, μ, M) then the vector representation of G(σ, μ, M) corresponds to I. If C is the symmetric difference between Ci and Cj then Ci ∩ Cj are counted twice in the sum of columns that corresponds to the edges of C.

Let C1, C2, . . . , Ck be the inclusion wise minimal fuzzy subsets of C. The columns corresponds to the edges of Ci. The sum of Ci corresponds to zero vector. Hence, Ci is disjoint and their union equals C.

The intersection of a circuit and a cocircuit in a fuzzy graphic matroid represents elements that are common to both the minimal dependent set and the minimal set whose removal in- creases the rank.

Theorem 3.5. If C is a circuit of a fuzzy graphic matroid and C∗ is a cocircuit, then the number of edges in (C ∩ C∗) is even.

Proof. Let G(σ, μ, M) be a fuzzy graphic matroid. C and C∗ be the circuit and cocircuit of G(σ, μ, M). If C ∩ C∗ = φ, it is obvious. Assume that the edge set E is contained in C and C∗ then there exists a base B∗ of G(σ, μ, M)∗ such that (C∗ − {μij}) ⊆ B∗ then the complementary base is B = E\B∗. Let $[I_r|D]$ be the vector representation of G(σ, μ, M) and assume {μij} ∈ B that corresponds to the first column of Ir and $[I_{n-r}|DT]$ be the vector representation of G(σ, μ, M)∗. C be the circuit of G(σ, μ, M), the columns of $[I_r|D]$ sums to zero vector. The odd number of columns of D have non zero entries in the first row. These columns correspond to the edges of C∗. (C − {μij}) and C∗ have odd number of common edges. Hence, (C − {μij}) ∩ C∗ has even number of edges.

The following lemmas are required to prove the theorem.

Lemma 3.6. If μij is an edge in the independent set of edges in a fuzzy graphic matroid

G(σ, μ, M), then there is a cocircuit C∗ such that C∗ ∩ I = {μij}.

Proof. B is a basis of G(σ, μ, M) ⊆ I. If B∗ is a basis of G(σ, μ, M)∗ that does not contain μij , B∗ ∪ {μij} is dependent, then there is a circuit C∗ of G(σ, μ, M)∗. Hence B∗ and B are disjoint therefore C∗ ∩ I = {μij}.

Lemma 3.7. Let P and Q be the subsets of edges of a finite set E. Every member of P is a member of Q and every member of Q contains a member of P. Then P and Q have same minimal members.

Proof. Let P be a minimal member of E. If E contain a member μjk of Q, Q contains a member of P that must be μij. Since P is minimal and μij is a member of Q. It implies that μij is a minimal member of Q. Similarly, μjk is a minimal member of P . Hence P and Q have the same minimal members.

The following theorems state the conditions needed for the fuzzy graphic matroid to be binary, which are based on the properties of a circuit in a fuzzy graph.

Theorem 3.8. For a fuzzy graphic matroid G(σ, μ, M), the following statements are equiva- lent:

1) G(σ, μ, M) is binary
2) If C is a circuit of fuzzy graphic matroid and C∗ is a cocircuit, then the number of edges in (C ∩ C∗) is even
3) If Ci and C_j are distinct circuits, then $C_i \Delta C_j$ is a circuit
4) If Ci and C_j are distinct circuits, then $C_i \Delta C_j$ is a disjoint union of circuits
5) The symmetric difference of any set of circuits is either empty or contains a circuit
6) The symmetric union of any set of circuit is a disjoint union of circuits

Proof. (i) ⇒ (iii) G(σ, μ, M) is binary and there is an isomorphism ϕ. If C_i and C_j are circuits of G(σ, μ, M),

$$\bigsqcup_{x\in \mathbf{C}_i} \phi(x) = 0 = \bigsqcup_{x\in \mathbf{C}_j} \phi(x)$$

$$\bigsqcup_{x\in \mathbf{C}_i} \phi(x) + \bigsqcup_{x\in \mathbf{C}_j} \phi(x) = 0$$

$C_i \Delta C_j = (C_i \cup C_j) - (C_i \cap C_j)$ each elements of $(C_i \cap C_j)$ counts double in $(C_i \cup C_j)$

$$\bigsqcup_{\mu_{ij}\in(\mathbf{C}_i\,\Delta\,\mathbf{C}_j)} = \bigsqcup_{\mu_{ij}\in\mathbf{C}_i} \phi(\mu_{ij}) + \bigsqcup_{\mu_{ij}\in\mathbf{C}_j} \phi(\mu_{ij}) - \bigsqcup_{\mu_{ij}\in(\mathbf{C}_i\,\cap\,\mathbf{C}_j)} \phi(\mu_{ij})$$

Therefore, $C_i \Delta C_j$ is dependent and contains a circuit.

(iii) ⇒ (ii) Assume G(σ, μ, M) does not satisfy (ii). The number of edges in (C ∩ C∗) is odd. If the elements in (C ∩ C∗) /= 1 then the edges in (C ∩ C∗) ≥ 3. Assume that the edges μij, μjk, μkl ∈ (C ∩ C∗) and μij, μkl ∈ C∗ then there exists a circuit that intersects C∗. The intersection of C and C∗ at E and μkl with (C ∩ (C∗ − {μjk}) that contains E.

Since μij ∈ (Ci ∩ C) and μjk ∈ (Ci − C). There exists (Cj ⊂ ((Ci ∪ C) − {μij}) containing

μjk. μjk ∈ (Cj ∩ C) and x ∈ (Cj − C). There exists Ck ⊂ C ∪ (Cj) − {μjk}) containing E.

$$(\mathbf{C} \cup \mathbf{C}_k) \subset (\mathbf{C} \cup \mathbf{C}_j) \subset (\mathbf{C} \cup \mathbf{C}_i)$$

Assume μ_{ij} ∈ Ck and μ_{ij} ∈/ Ck then C ∪ Ck is minimal.

$$(\mathbf{C}_j - \mathbf{C}) = (\mathbf{C}_k - \mathbf{C})$$

Therefore C_j /= C_k. Assume that $(C_j \Delta C_k)$ contains a circuit. $(C_j \Delta C_k)$ = C hence $(C_j \cap C^*)$ and $(Ck \cap C^*)$ are non-empty. μij ∈ $(C_k \cap C^*)$ and μjk ∈ $(C_j \cap C^*)$ are smaller than $(C \cap C^*)$. The number of edges in $C \cap C^*)$ is odd but the number of edges in (Ck ∩ C∗) and (Cj ∩ C∗) are even. This is a contradiction to our assumption. Therefore, (ii) holds.

(ii) ⇒ (v) Let Ci, Cj, . . . , Cn be the circuits of fuzzy graphic matroid. Then Ci Δ CjΔ . . . ΔCn is non empty and independent. Let μij ∈ Ci Δ Cj . . . ΔCn by Lemma 3.7 there exists a cocir- cuit C∗ such that

$$\mathbf{C}* \cap (\mathbf{C}_i \,\Delta\, \mathbf{C}_j\Delta \ldots \Delta\mathbf{C}_n) = \mu_{ij}$$

The edges in (C∗ ∩ C) is even by (ii). C∗ ∩ (Ci Δ Cj Δ . . . Δ Cn) is even which is a contradiction. Hence, (v) holds.

The results (iv) ⇒ (iii) and (vi) ⇒ (iv) are obvious.

(v) ⇒ (vi) Assume C_i Δ C_jΔ . . . Δ Cn is not a disjoint union of circuits of G(σ, μ, M). Considering C with minimum number of edges, the symmetric difference of the circuits of

G(σ, μ, M) is non empty. From (v), C_i Δ C_j Δ . . . Δ Cn contains a circuit, then C_i Δ C_j Δ . . . Δ Cn

has a minimum number of edges and so it is a disjoint union of circuits.

Theorem 3.9. If G(σ, μ, M) be a fuzzy graphic matroid, then the following statements are equivalent:

1) G(σ, μ, M) is binary
2) C and C∗ are the circuits and cocircuits of G(σ, μ, M), then the number of edges in

(C ∩ C∗) /= 3

3) G(σ, μ, M) has no minor which is isomorphic to U2,4

Proof. (i) ⇒ (iii) From Theorem 3.3, G(σ, μ, M) is binary if and only if U2,4 is not its minor.

(i) ⇒ (ii) From Theorem 3.5, If C is a circuit of a fuzzy graphic matroid and C∗ is a cocircuit, then the number of edges in (C ∩ C∗) is even.

(ii) ⇒ (iii) Assume that the fuzzy graphic matroid G(σ, μ, M) has a minor isomorphic to

U2,4. Let S be a 3-element fuzzy subset of I and the maximal independent set of edges in I has 2 edges. If there exist a circuit and cocircuit of G(σ, μ, M) then their intersection is S. Hence, the edges in (C ∩ C∗) is 3 which is a contradiction and so the number of edges in

(C ∩ C∗) /= 3.

4. CONSTRUCTION OF NETWORKS

A method for constructing a network with matroids is introduced [15]. A map which maps the fuzzy matroid to the network such that the obtained network has vector representation over the field if and only if the fuzzy matroid has multilinear representation over the same field. This construction gives a strong relation between the network and fuzzy matroid theory.

Let G be a fuzzy graph having n vertices and m edges. G is a pair of functions σ∗ :

V −→ [0, 1] and μ∗: V × V −→ [0, 1]. Define n = |E| is the total number of messages. The communication network defined from a pair N (G(σ∗, μ∗), f) from a fuzzy graph G and a function f : γ ∪ E is the set of messages qi = (qi1, qi2, . . . , qin). The ends of the edges μi(i = 1, 2, . . . , n) holds a messages qi. The edges of the fuzzy graph represents the communication links that transmits messages. The main objective is to identify an encoding scheme known as index code that satisfies the demand of all receivers with the minimum number of transmissions [4].

A fuzzy graph G involved in the formulation of index coding problem, index codes and net- work codes are strongly related. An index code over a finite field Fq of length l and dimension n for the ICP is a function f : Fm −→ Fl .

The additional information and the demands of the receivers are the functions of messages rather than a subset of messages. The additional information possesses by the receivers is

described by a Has − set, which consists of function of messages [17]. The demands of the

receivers are described by Want − set. Each receiver Wi is described by a tuple (Wi, Hi), where Wi, Hi are the set of function from Fm −→ Fl

q q

The index coding problem ICP [M, W] includes

1) A set of q messages {qi} where i = 1, 2, . . . , n
2) A set of receivers W ⊆ {q, C}; q ∈ C, C ⊆ W\{μ_{ij}}

Here, qi represents the set of messages. Each message qi belongs to certain alphabet f}n. A receivers is represented by a pair (q, C) and C ⊆ M is the set of messages available as infor- mation. Assume the message qi can be divided into n packets qi = (qi1, qi2, . . . , qin).

Consider the index coding problem for finding an optimal index code for an index coding instance. Given instance a, b the ICP (M, W) of the index coding problem is defined. Let μ(ICP) be the maximum of the total number of messages requested by a set of receivers. Let ICP [M, W] be an instance of an index coding problem. An index code for ICP [M, W] achieving λ(a, b) = μ(ICP) is defined to be the perfect index code.

4.1. CONVERSION FROM FUZZY GRAPHIC MATROID TO INDEX CODE

The fuzzy graphic matroid is said to have multilinear representation of dimension n or an n-linear representation over a field F. The construction of an instance captures important properties from fuzzy matroids. A fuzzy graphic matroid G(σ, μ, M) of rank R over a ground set E, the corresponding index coding problem ICP [M, W] is described as follows:

1) G(σ, μ, M) = Y ∪ X, where X = {x1, x2, . . . , xn}
2) W = R1 ∪ R2 ∪ R3 where
   - R1 = {(x_i, B); i = 1, 2, . . . , n}
   - R2 = {(y, C\{y}); y ∈ C}
   - R3 = {(y_i, I); i = 1, 2, . . . , m}

Let (a, b) is a linear code for ICP [G(σ, μ, M), W]. The existence of the decoding function of receivers in W is shown as follows:

a) Fix a fuzzy basis B = {x_{i1}, x_{i2}, . . . , x_{ik}} ∈ B, with i1 < i2 < . . . < $_{ik}$ and let E_i = (x_i, B) ∈ R_1, i = 1, 2, . . . , n. The corresponding decoding function can be written as $\pi E_i = [f_{i1} - y_{i1}| \dots |f_{ik} - y_{ik}]U_i$.

b) Let C = {y_{i1}, y_{i2}, . . . , y_{ik}} ∈ B, with i1 < i2 < . . . < ik and let E = (y_{i1}, FC/) ∈ R2 with

C = C− Y_{i1}. The corresponding decoding function $\pi Ei = [f_{i1} - [f_{i2} - y_{i2}| \dots |f_{ik} - y_{ik}]A$

c) E = (y_i, A) ∈ R3

Let B is a basis of fuzzy graphic matroid then by (b) the Receivers (x_j, B), j = 1, 2, . . . , n be able to decode their required messages and C be a circuit of fuzzy graphic matroid. y_i ∈ C and C/ = C − yi1 then R(C/) = |C| − 1 = |C/ |. C/ is an independent set of fuzzy graphic matroid and there is a basis B of G(σ, μ, M). Consider the receiver W = (yi1, FC/) ∈ R2, the existence of linear decoding function $\pi_{�}$ implies that there exists a matrix A.

4.2. CONSTRUCTION OF NETWORK USING INDEX CODES

Let N be a network with message set γ and edge set E. Let G(σ, μ, M) = (E, I) is a fuzzy graphic matroid with rank function R. The network N is associated with fuzzy graphic matroid G(σ, μ, M) there exists a function f : γ ∪ E −→ E such that f is a network to fuzzy graphic matroid mapping.

For a fuzzy graphic matroid, the construction of a network is defined as follows:

1) Create a network source and corresponding messages qi1, qi2, . . . , qin. Choose a base

B = {B_1, B_2, . . . , B_n} of G(σ, μ, M).

2) Find a circuit C1, C2, . . . , Cn in G(σ, μ, M) and add a source with a message to the network. A new receiver demands message.

4.3. ALGORITHM FOR CONSTRUCTING OF NETWORKS FROM INDEX CODES

For a fuzzy graphic matroid G(σ, μ, M) = (E, I) having rank R, the fuzzy set is given by $E = \{q_1, q_2, \dots, q_n\}$ and ICP [G(σ, μ, M), W] is associated with the fuzzy graph G the step-wise procedure to construct a network is given below:

Step (i) Let G(σ, μ, M) be a fuzzy graphic matroid with rank R, then the matrix representation of G(σ, μ, M) is given by R×n. Let $E = \{x_i, \dots, x_n\} \in FR$ and $\gamma = \{x_i, \dots, x_n, y_i, \dots, y_n\} \in F_2^{n+R}$

Step (ii) The mapping f (γ) = (I(γ), . . . , fm(γ)) where

$I(\gamma) = xi + \lambda(FM) \in Fn+R, i = 1, 2, \dots, n$

Step (iii) The receivers are able to satisfy their demands using their Has− sets and the transmitted messages.

- Receivers in W1: Consider the fuzzy bases $B = \{B_1, \dots, B_k\} \in B(G(\sigma, \mu, M))$ and let $(xi, B) \in W_1$.

$[fi1 (\gamma), \dots, fik (\gamma)] = [x_{i1}, x_{i2}, \dots, x_{ik}]+\lambda[G(\sigma, \mu, M)_{i1}, G(\sigma, \mu, M)_{i2}, \dots, G(\sigma, \mu, M)_{ik}]$

since $\{\mu_{i1}, \mu_{i2}, \dots, \mu_{ik}\} \in Q(G(\sigma, \mu, M))$ the matrix obtained from $G(\sigma, \mu, M)_{i1}$,
G(σ, μ, M)i2 , . . . , G(σ, μ, M)ik is invertible. The receivers can obtain λ using this rela- tion
$\lambda = [fi (\gamma) - \mu_i\ fi (\gamma) - \mu_i \dots fi (\gamma) - \mu_i]\ B-1$

- Receivers in W2 :Let $C = \{\mu_{i1}, \mu_{i2}, \dots, \mu_{ik}\} \in C(G(\sigma, \mu, M))$ and $(\mu_i, C) \in W_2$.

$[f_{i2} (\gamma), \dots, f_{ik} (\gamma)] = [\mu_{i2}, \mu_{i3}, \dots, \mu_{ik}]+\lambda[G(\sigma, \mu, M)_{i2}, G(\sigma, \mu, M)i3, \dots, G(\sigma, \mu, M)_{ik}]$
All the receivers in W2 can decode their demanded messages by

$\mu_{i1} = (f_{i1} (\gamma) + f_{i2} (\gamma) + \dots + f_{ik} (\gamma)) + (\mu_{i1} + \mu_{i2} + \dots + \mu_{ik})$

- Receivers in W3 : (μi, I) ∈ W3, receivers can obtain their demanded messages using the relation $\mu i = I(\gamma) - \lambda(G(\sigma, \mu, M)_i)$

Step (iv) The ICP (X, W) is associated to the network N (ICP) over the fuzzy graph G is con- structed using the index codes.

5. IMPLEMENTATION, RESULTS AND DISCUSSION WITH AN EXAMPLE

A simple network is constructed using the generation of index codes from fuzzy graphic matroids. There are three receivers W1, W2 and W3 with a message set γ and additional messages s1 and s2. The messages are coded and decoded using the constructed algorithm.

5.1. RESULTS AND DISCUSSION

Initial work on nework coding focused on multicast network for which maximum number of packets sent from the source S per unit time equals the minimum capacity of all cuts seperating the source S. The proposed algorithm is different from the already existing algorithm in this following way:

The network from an index coding problem associated with a fuzzy graphic matroid consists of edges representing all the messages available at the transmitter and the edges corresponding to the receivers are the availability of information captured by the edges connecting a receiver to the corresponding source carrying the information. The noiseless channel is modeled using fuzzy graphic matroid by a set of edges connected to all the messages and receivers. The advantage of this research work over other work lies in the generation of index codes using fuzzy graphic matroid and it is decoded to obtain the messages from source to the receivers. The network is constructed using the transmission of messages qi.

5.2. IMPLEMENTATION WITH AN EXAMPLE

The authors constructed a basic network using index codes from fuzzy graphic matroids and solved a numerical problem using the proposed algorithm.

Step (i) Let G(σ, μ, M) = (E, I) be a fuzzy graphic matroid defined on a ground set E = $\{(\mu_{i1}, 0.4), (\mu_{i2}, 0.2), (\mu_{i3}, 0.3)\}$ with rank function. The ICP with side information cor- responding to the fuzzy graphic matroid has the message set γ = $\{(\mu_{i1}, 0.4), (\mu_{i2}, 0.2), (\mu_{i3}, 0.3), (\mu_{i4}, 0.2), (\mu_{i5}, 0.4)\}$, where each message belongs to the finite field F2.

Step (ii) The ICP is given by the mapping $f: F_2^5 \longrightarrow F_2^3$

f (γ) = $\{(\mu_{i1}, 0.4), (\mu_{i2}, 0.2), (\mu_{i3}, 0.3)\} + \{(\mu_{i4}, 0.2), (\mu_{i5}, 0.4)\}$ FGM

Step (iii) The set of receivers is given by

- Receivers in W_1: $\{(\mu_{i4}, 0.2), \{(\mu_{i1}, 0.4), (\mu_{i2}, 0.2)\}\}, \{(\mu_{i5}, 0.4), \{(\mu_{i1}, 0.4), (\mu_{i2}, 0.2)\}\}, \{(\mu_{i4}, 0.2), \{(\mu_{i1}, 0.4), (\mu_{i5}, 0.3)\}\}, \{(\mu_{i4}, 0.4), \{(\mu_{i1}, 0.4), (\mu_{i3}, 0.3)\}\}, \{(\mu_{i5}, 0.2), \{(\mu_{i2}, 0.2), (\mu_{i3}, 0.3)\}\}, \{(\mu_{i5}, 0.4), \{(\mu_{i2}, 0.2), (\mu_{i3}, 0.3)\}\}$
- Receivers in W_2:$\{(\mu_{i1}, 0.4), \{(\mu_{i2}, 0.2) + \mu_{i3}0.3\}\}, \{(\mu_{i2}, 0.2, \{(\mu_{i3}, 0.2) + (\mu_{i1}, 0.4)\}\}, \{0.3, \{(\mu_{i2}, 0.2) + (\mu_{i1}, 0.4)\}\}$
- Receivers in W_3:$\{(\mu_{i1}, 0.4), \{(\mu_{i4}, 0.2), (\mu_{i5}, 0.4)\}\}, \{(\mu_{i2}, 0.2), \{(\mu_{i4}, 0.2), (\mu_{i5}, 0.4), 0.4\}\}, \{(\mu_{i3}, 0.3), \{(\mu_{i4}, 0.2), (\mu_{i5}, 0.4)\}\}$

The index codes for W_1, W_2 and W^3 are

$c_1 = \{0.4 + 0.2\} = 0.6$

$c_2 = \{0.2 + 0.4\} = 0.6$

$c_3 = \{0.3 + 0.2 + 0.4\} = 0.9$

The decoding of the index codes of W_1, W_2 and W_3 are

Receivers in W_1	Decoding
$\{(\mu_{i4}, 0.2), \{(\mu_{i1}, 0.4), (\mu_{i2}, 0.2)\}\}$	1.0
$\{(\mu_{i5}, 0.4), \{(\mu_{i1}, 0.4), (\mu_{i2}, 0.2)\}\}$	0.8
$\{(\mu_{i4}, 0.2), \{(\mu_{i1}, 0.4), (\mu_{i3}, 0.3)\}\}$	1.0
$\{(\mu_{i5}, 0.4), \{(\mu_{i1}, 0.4), (\mu_{i3}, 0.3)\}\}$	2.2
$\{(\mu_{i4}, 0.2), \{(\mu_{i2}, 0.2), (\mu_{i3}, 0.3)\}\}$	2.0
$\{(\mu_{i5}, 0.4), \{(\mu_{i2}, 0.2), (\mu_{i3}, 0.3)\}\}$	0.8

Receivers in W_2	Decoding
$\{(\mu_{i1}, 0.4), \{(\mu_{i2}, 0.2) + \mu_{i3}, 0.3\}\}$	2.6
$\{(\mu_{i2}, 0.2), \{(\mu_{i1}, 0.4) + \mu_{i3}, 0.3\}\}$	2.8
$\{(\mu_{i3}, 0.3), \{(\mu_{i1}, 0.4) + \mu_{i2}, 0.2\}\}$	2.7

Receivers in W_3	Decoding
$\{(\mu_{i1}, 0.4), \{(\mu_{i4}, 0.2), (\mu_{i5}, 0.4)\}\}$	0.8
$\{(\mu_{i2}, 0.2), \{(\mu_{i4}, 0.2), (\mu_{i5}, 0.4)\}\}$	1.0
$\{(\mu_{i3}, 0.3), \{(\mu_{i4}, 0.2), (\mu_{i5}, 0.4)\}\}$	1.5

Step (vi) The network constructed from the index codes is given in Figure 5.1

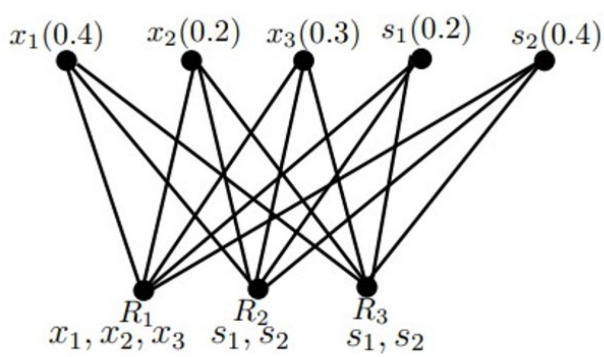

Figure 5.1 The Network constructed using Index codes

6. CONCLUSION

In this paper, the authors established a method to construct a fuzzy matroidal network using fuzzy graphic matroids. Also, presented a technique for constructing a network such that any multilinear representation of a fuzzy graphic matroid that naturally leads to a linear network code over the same field. A multilinear representation of fuzzy graphic matroids extends from vector space −→ are given. Furthermore, the application of index codes in modelling the creation of a noiseless channel network through a set of edges connecting the message set qi to the receivers are explored. The authors further proposed to work on error corrections in the index codes using fuzzy graphic matroids.

CONFLICT OF INTERESTS

None.

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