Original Article
ISSN (Online): 2582-7472

ShodhKosh: Journal of Visual and Performing Arts
July-December 2021 2(2), 281–288

PAPER ON ANALYTICAL SOLUTION OF HIGHER ORDER PARTIAL DIFFERENTIAL EQUATIONS

Chetan Rathod [1]

[1] Lecturer, Department of Mathematics, Swami Vivekananda Degree College, Mudddebihal-586212

Corresponding Author
Chetan Rathod,
chetanrathod8777@gmail.com

DOI
10.29121/shodhkosh.v2.i2.2021.3654

Funding: This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

ABSTRACT

For a long time, detachment of variable is perceived as one of the most remarkable strategies for settling straight incomplete differential conditions PDEs. The current paper proposes scientific answer for higher request homogeneous fractional differential conditions PDEs under determined limit conditions BCs inside a rectangular space. Partition of factors and basic variables, first and foremost, are utilized to decrease the given incomplete differential condition PDE to a common differential condition Tribute. After representative controls, a power series development of the obscure capability is used to make the logical arrangement. The current paper is an extraordinary instance of partition of factors which depend on dispensing with one variable to settle the PDE on the other variable. The proposed shut structure arrangement introduced here lessens the work consumed for carry out the option mathematical arrangements. The viability of the got strategy demonstrates the ability to give a scientific arrangement beating the intricacy of limit conditions and blended subordinates in the arrangement of higher request straight PDE

Keywords: Partial Differential Equation, Separation of Variables, Shape Function, Power Series

1. INTRODUCTION

Through ongoing many years, different execution of higher request direct PDEs was utilized, alongside the observational procedures, in shape plan for designing assembling, or as a rule, for the portrayal of strong surfaces. In computer-aided geometric design, surface/solid models are represented and manipulated with PDEs. They possessed major significance in designing for examination and recreation notwithstanding their significance in clinical area for body tissue perception and careful reproduction [2]-[4]. Accurate arrangement of direct PDEs of many designing applications still as a primary concern the mathematicians and particular distributers. M.H. Martin [5] observed a special case of a two-term equation that could be derived from Laplace's equation by separating variables at the beginning of the 1950s. His documentation laid out a defining moment for the conventional techniques for decrease, opening up new dreams to what is currently known as utilitarian detachment of factors. The higher request multi-term PDE including partial subsidiary in time is concentrated by E. Karimov and S. Pirnafasov [6]. They utilized detachment of factors to diminish partial request PDE to the number request. I.V. Rakhmelevich [7] changed complex PDE consolidating straight differential administrator to diminished one and settled it by utilizing partition of variable. W. N. Everitt and B. T.

Johansson [8] solved the Dirichlet problem for the biharmonic PDE on a bounded region using quasi-separation of variables. A. S. Berdyshev and B. J. Kadirkulov [9] read up a nonlocal issue for a fourth-request explanatory condition with the Dzhrbashyan- Nersesyan partial differential administrator. They utilized detachment of factors to demonstrate the presence and uniqueness of the arrangement of the issue. Various approaches, such as the Fourier series and the Galerkin formulation, have already been used by publishers to attempt to separate variables of higher order PDE.

Profoundly precise scientific answers for some applications, for example, anisotropic and orthotropic rectangular plates are presented by a few specialists [10]-[12]. A semi-analytic approach based on applying a circumvention between integral factors and generalized variable separation is proposed in this paper. The inferred procedure prompts careful shut structure answer for higher request PDEs integrating blended subsidiaries. The current technique is applied to higher request direct PDEs with steady coefficients. The shape capability is made by the examined PDEs and limit conditions. On the other hand, the power series polynomial is how the unknown separated function is expressed. Under the proposed boundary conditions BCs, the derived method is used to establish the closed form solution of composite plate vibration.

2. PARTIAL DIFFERENTIAL EQUATION PDE

The general higher order homogenous PDE is:

$$f\left(x,y,w,\frac{\partial^n w}{\partial x^n},\frac{\partial^n w}{\partial x^r \partial y^{n-r}},\dots\dots,\frac{\partial^n w}{\partial y^n},\frac{\partial^{n-1} w}{\partial x^{n-1}},\frac{\partial^{n-1} w}{\partial x^r \partial y^{n-r-1}},\dots\dots,\frac{\partial^{n-1} w}{\partial y^{n-1}},\frac{\partial^{n-2} w}{\partial x^{n-2}},\frac{\partial^{n-2} w}{\partial x^r \partial y^{n-r-2}},\dots\dots,\frac{\partial^{n-2} w}{\partial y^{n-2}},\dots\dots,\frac{\partial w}{\partial x},\frac{\partial w}{\partial y}\right)=0; \quad r=1,2,3,\dots\dots\dots,n-1 \tag{1}$$

Where

w = w(x, y)is the unknown function of the dependent variables x, y .

Let us consider the dimensionless PDE over a rectangular r region (a ×b) ∈ $\Re$ in the short form:

$$(Lw)(\zeta,\eta) = 0 \tag{2}$$

where L denotes the linear partial differential operator and w = w (ζ ,η) is the dimensionless unknown function where

$$\zeta = \frac{x}{a} \ and \ \eta = \frac{x}{b}$$

Separation of variables assumes general solution of the form:

$$w(\zeta,\eta) = \sum_{m=1}^{M} g_n(\zeta)\, f_m(\eta) \tag{3}$$

Where $g_n(\zeta)$ and $f_m(\eta)$ are functions satisfying the boundary conditions of rectangular region at the boundaries (ζ= 0,1) and $(\zeta = 0,1)$respectively. Equation (3) is utilized to reduce Eq. (2) to the ordinary differential equation ODE:

$$\sum_{n=0}^{N} c_{mn} f^{(n)}(\eta) = 0 \tag{4}$$

where the function $f(n)(\eta)$ is unknown function differentiated with order n while Cmn is integrated constant based on the known function $gm(\zeta)$ so that:

$$c_{mn} = \int_0^1 g_m\, {g^{(n)}}_m d\eta \tag{5}$$

Series solution of the reduced ODE (4) is:

$$w(\zeta,\eta) = \sum_{m=1}^{M} g_m(\zeta)\left[f_m(0) + \sum_{k=1}^{K} {f^{(k)}}_m(0)\frac{\eta^k}{k!}\right] \tag{6}$$

where

$f_m{}^{(k)}(0) = \frac{d^k f}{d\eta^k}$ at $\eta = 0; k = 1,2,3 ....$ (7)

Consequently:

$$w(\zeta,\eta(= \sum_{m=1}^{M} g_m(\zeta)\left\{f_m(0) + f'_m(0)\eta + f''_m(0)\frac{\eta^2}{2!} + f'''_m(0)\frac{\eta^2}{3!} + \cdots ..... + f^{(n-1)}{}_m(0)\frac{\eta^{n-1}}{(n-1)!} + n(0)\frac{\eta^n}{n!} + E_p\right\} \quad (8)$$

Where,

$$E_0 = \sum_{k=n}^{K} f^{(k)}{}_m(0)\frac{\eta^k}{k!} \quad (9)$$

The function E0 is a truncated power series function, constructing on the constant coefficients 〚f^((k))〛_m $f^{(k)}{}_m(0); k = n, n+1, n+2 \dots \dots .. K$ K which epend on the n initial values

$f_m(0), f'_m(0), f''_m(0), f'''_m(0), \dots \dots \dots f^{(n-1)}{}_m(0)$ so that:

$$f_m{}^{(k)}(\eta) = \frac{d}{d\eta} f_m{}^{(k-1)}(\eta); k = n+1, n+2 \dots \dots .. K \quad (10)$$

Where $f_m{}^{(n)}(\eta)$ is is defined according to Eq.(4).

Initial values f_m (0), $f_m(0), f'_m(0), f''_m(0), f'''_m(0), \dots \dots \dots f^{(n-1)}{}_m(0)$_m (0) are accomplished from the known boundary condition at (η=0,1).

Shape Function (〚(ζ)〛 _m g and Seperation Of Variables

Partition of factors is applied to make the shape capability in only one variable of straight fractional differential conditions under limit and starting circumstances utilizing

$$w(\zeta,\eta) = g(\zeta)f(\eta) \quad (11)$$

For cases which don't include blended subordinates, for example, the intensity condition, Laplace condition, Helmholtz condition or wave condition, partition of factors is effectively being finished. Be that as it may, for cases including blended subordinates, for example, bi harmonic condition , the PDE isn't effortlessly isolated, yet regardless Eq (11) may in any case be applied. As indicated by division, two single variable frameworks of Tributesare acquire under limit conditions. Fulfilling the proposed limit conditions, shape capability is straight forwardly characterized. For instance, the dimensionless bi harmonic condition is thought of :

$$\nabla^2 w = 0 \text{ where } \nabla = \frac{\partial^2 w}{\partial \zeta^2} + \tau^2 \frac{\partial^2 w}{\partial \zeta^2} + \tau^2 \frac{\partial^2 w}{\partial \eta^2} \quad (12)$$

Where $\tau = \frac{a}{b}$ is the aspect ratio of the rectangular region.

Subsection from (11) in (12), gives:

$$\frac{g^{(4)}(\zeta)}{g(\zeta)} + 2\tau^2 \frac{g''(\zeta)}{g(x)}\frac{f''(\eta)}{f(\eta)} + \tau^4 \frac{f^{(4)}(\eta)}{f(\eta)} = 0$$

Rewriting Eq. (13) in the form:

$$G(\zeta) + 2\tau^2 E(\zeta)H(\eta) + \tau^4 F(\eta) = 0 \quad (14)$$

Eliminating the first and last terms by differentiating Eq. (14) w.r.t ζ and η; yields:

$$E'(\zeta)H'(\eta) = 0$$

This means that either E(ζ) or H(η) must be a constant , say μ_1 or μ_2. Consequently,

$-G(\zeta) = 2\tau^2 F(\zeta)H(\eta) + \tau^4 F(\eta)$ or $-\tau^4 F(\eta) = G(\zeta) + 2\tau^2 E(\zeta)H(\eta)$ is constant.

Differentiating all terms w.r.t $(\zeta)$and (η), one can prove that each of $G(\zeta)$and $F(\eta)$ are constant, say $\omega_1$or ω_2. Thus two cases are obtained:

$$f^{(4)}(\eta) + 2\mu_1\tau^2 f^{''}(\eta) + \tau^4\omega_1 f(\eta) = 0 \qquad (15)$$

And $g^{(4)}(\zeta) + 2\mu_2\tau^2 g^{''}(\zeta) + \tau^4\omega_2 g(\zeta) = 0$ (16)

Where

$$\frac{g^{''}(\zeta)}{g(\zeta)} = \mu_{1,} \frac{g^{(4)}(\zeta)}{g(\zeta)} = \omega_1 \quad , \quad \frac{f^{''}(\eta)}{f(\eta)} = \mu_{2,} \frac{f^{(4)}(\eta)}{f(\eta)} = \omega_2 \qquad (17)$$

Each case is homogenous ODE which can easily be solved by auxiliary equation. Because of $g^{(4)}(\zeta) = \omega_1 g(\zeta)$ and $g^{''}(\zeta) = \mu_1 g(\zeta)$, then : $g^{(4)}(\zeta) = \mu_1 g''(\zeta)$. Thus, one can prove that $\omega_1 = {\mu^2}_1$. Similarly $\omega_2 = {\mu^2}_2$

The general solution of equation (16), is the shape function g_m (ζ) which is in the form:

$$g_m(\zeta) = A_1 \sin(\alpha_m\zeta) + A_2 \cos(\alpha_m\zeta) + A_3 \sinh(\alpha_m\zeta) + A_4 \cosh(\alpha_m\zeta) \qquad (18)$$

Where α_m and A_1,A_(2,) A_(3,) A_4 are constant parameters depending on τ,μ_2 and boundary conditions at the edges of support at (ζ=0,1). Generalized techniques for separation of variables have been described by many researches in literature [13],[14].

3. APPLIED STUDY

Solution of PDE of Composite Plate

To outline and inspect the current procedure, a utilization of square composite plate is given. The review will focus on the accompanying fourth request PDE for vibration of a composite plate:

$$C_0 w_{\zeta\zeta\zeta\zeta} + C_1\tau w_{\zeta\zeta\zeta\eta} + C_2\tau^2 w_{\zeta\zeta\eta\eta} + C_3\tau^3 w_{\zeta\eta\eta\eta} + C_4\tau^4 w_{\eta\eta\eta\eta} + \rho h a^4 w_{tt} = 0 \qquad (19)$$

Where $w = w(\zeta,\eta,t)$is the plate dimensionless displacement, the subscripts denote to partial derivatives with respect to the independent variables, ζ,η,t where ζ,η denote the dimensionless coordinates such as $\zeta = \frac{x}{a}$,$\eta = \frac{y}{b}$while t denotes time. The magnitude τ is the aspect ratio where $\tau = \frac{a}{b}$ and a,b are he dimensions of plate in Cartesian x,y directions respectively. The constant coefficients $C_n, n = 0,1,2,3,4$are composite plate parameters while ρ is plate mass per unit volume and h is the thickness of plate. The solution is assumed to be:

$$w(\zeta,\eta,t) = \sum_{m=1}^{M} g_m(\zeta) f_m(\eta) e^{i\omega t} \qquad (20)$$

Where $g_m(\zeta)$ is knoen shape function satisfying the boundary conditions of plate at the two edges (ζ=0,1).

The assumed form (20) is used to reduce Eq. (19) to :

$$\sum_{m=1}^{M}\left[\alpha_m C_4\tau^4 {f^{'''}}_m(\eta) + \beta_m C_3\tau^3 {f^{''}}_m(\eta) + \gamma_m C_2\tau^2 {f^{''}}_m(\eta) + \rho_m C_1\tau {f^{'}}_m(\eta) + (q_m C_0 - \omega^2 p h a^4\alpha_m) f_m(\eta)\right] = 0 \qquad (21)$$

Where

$$\alpha_{mn} = \int_0^1 g_m g_m d\zeta, \beta_{mn} = \int_0^1 g_m {g^{'}}_m d\zeta, \gamma_{mn} = \int_0^1 g_m g^{''} d\zeta, p_{mn} = \int_0^1 g_m {g^{'''}}_m d\zeta, q_{mn} = \int_0^1 g_m {g^{'''}}_m d\zeta \qquad (22)$$

And

$${f^{(4)}}_m(\eta) = -\frac{\beta_m C_3}{\alpha_m\tau C_4}{f^{'''}}_m(\eta) - \frac{\gamma_m C_2}{\alpha_m\tau^2 C_4}{f^{''}}_m(\eta) - \frac{p_m C_1}{\alpha_m\tau^3 C_4}{f^{'}}_m(\eta) - \frac{(q_m\frac{C_0}{4} - \lambda^2\alpha_m}{\alpha_m\tau^4} f_m(\eta) \qquad (23)$$

The initial values $f_m(0), {f^{'}}_m(0), {f^{''}}_m(0), {f^{'''}}_m(0)$are defined due to known boundary condition at (η=0,1). Consequently, the derivatives ${f^{(k)}}_m(0); k = 4,5,6 \ldots \ldots \ldots K$ are achieved according to :

$f_m{}^{(k)}(\eta) = \frac{d}{d\eta} f_m{}^{(k-1)}(\eta); k = 5,6,7, \dots \dots \dots . K$ (24)

The final solution expressed in terms of the k^th power degree of η is:

$$w(\zeta,\eta) = \sum_{m=1}^{M} g_m(\zeta) \Big\{ f_m(0) + f_m{}'(0)\eta + f_m{}''(0)\frac{\eta^2}{2!} + f_m{}'''(0)\frac{\eta^3}{3!} - \frac{1}{B_4}\left[B_3 f_m{}'''(0) + B_2 f_m{}''(0) + B_1 f_m{}'(0) + B_0 f_m(0)\right]\frac{\eta^4}{4!} - \frac{1}{B_4}\,[-\frac{B_3}{B_4}[B_3 f_m{}'''(0) + B_2 f_m{}''(0) - \frac{1}{B_4}[-\frac{B_3}{B_4}[-\frac{1}{B_4}B_3 f_m{}'''(0) + B_2 f_m{}''(0) + B_1 f_m{}'(0) + B_0 f_m(0) - \frac{B_2}{B_4}\left[B_3 f_m{}'''(0) + B_2 f_m{}''(0) + B_1 f_m{}'(0) + B_0 f_m(0) - \frac{B_2}{B_4}\left[B_3 f_m{}'''(0) + B_2 f_m{}''(0) + B_1 f_m{}'(0) + B_0 f_m(0)\right] + B_1 f_m{}'''(0) + B_0 f_m{}''(0)\right]\frac{\eta^6}{6!} + \cdots \dots \dots . + f_m{}^{(k)}(0).\frac{\eta^k}{k!}\Big\} \quad (25)$$

Where,

$$B_0 = \left(q_m\frac{C_0}{C_4} - \lambda^2{}_m\alpha_m\right), B_1 = p_m C_1\tau, B_2 = \gamma_m C_2\tau^2, B_3 = \beta_m C_1\tau^3, B_4 = \alpha_m C_4\tau^4 \; and \; \lambda^2{}_m = \omega^2 a^4\frac{\bar{m}}{C_4}, \bar{m} = \rho h$$

Numerical Results

The case of vibration of a square simply supported SSSS arthotropic plate is examined here in order to realize the explicit and implicit solutions derived from equation (25) under specific boundary conditions. An orthotropic carbon-epoxy plate has a thickness of 0.75 millimeters, and its material properties and bending stiffnesses are as follows [15]:

D11=409.08, D22=23.28,D12=6.52, D16=D26=0 and D66=16.13

The eigen valuesλ_mn of this case are, implicity/ explicity, obtained for truncation number k= 12 of the power series. Three modes of vibrations are examined showing the following results:

First Mode m=1

The explicit eigen values $\lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14}, \dots\dots \lambda_{1n},$ when m=1 are obtained from the solution of the following algebraic equation :

$1.8092214146(10^{19}) - 2.015964353(10^{16})\lambda_1{}^2 + 8.920238451(10^{12})\lambda_1{}^4 - 2.206726296(10^9)\lambda_1{}^6 + 2.623401997(10^5)\lambda_1{}^8 = 0$ (26)

Thus, the first three values of λ_1n are

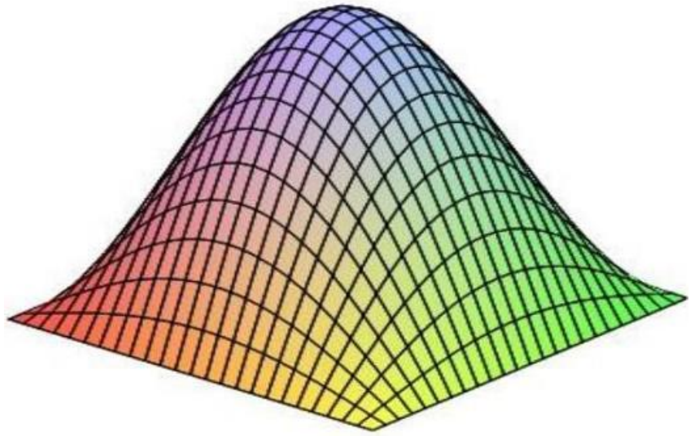
$$\lambda_{11} = 44.71021071, \lambda_{12} = 56.24689473, \lambda_{13} = 49.63224921$$

To illustrate the effectiveness of the present technique, the eigen function (mode shape) corresponding to an eigen value, say $\lambda_{11} = 44.71021071$, is expressed in explicit closed form:

$w_{11}(\zeta,\eta) = [1.0(10)^9\zeta + .5\zeta^2 - 1.646275758(10)^9\zeta^3 + .8002095248\zeta^4 + 8.134474111(10)^8\zeta^5 + 0.1478451609\zeta^{10} - 2.341320677(10)^6\zeta^{11}][\sin \pi\eta] = 0$ (27)

This mode shape ids visualized in Fig.1.

Fig.1. Plate First Mode Shape, λ_11=44.71021071,m=1,n=1



2.Second Mode m=2

Similarly, the explicit eigen value $\lambda_{21},\lambda_{22},\lambda_{23},\lambda_{24},\ldots\ldots\lambda_{2n}$ are expressed in :

$$2.079942336(10)^{23}-2.769390049(10)^{19}\lambda_2^{2}+1.388812471(10)^{15}\lambda_2^{4}-3.109683007(10)^{10}\lambda_2^{6}+2.622340201(10)^{5}\lambda_2^{8}=0 \quad (28)$$

Fig.2. Plate Second Mode Shape,

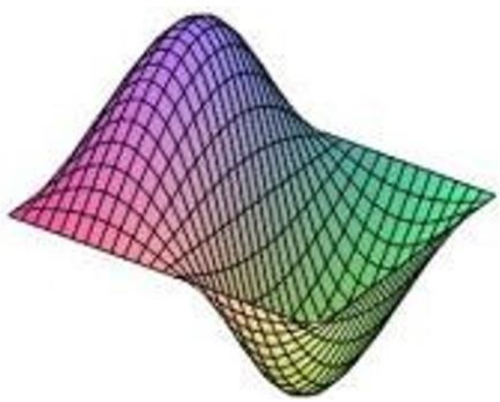


Figure 2 Plate Second Mode Shape, $\lambda_{21} = 168.05635479, m = 2, n = 1$

Consequently, the first three values of λ_{2n} are

$$\lambda_{21} = 168.05635479, \lambda_{22} = 173.0464903, \lambda_{23} = 175.6330584$$

Also, the mode shape of plate corresponding to:

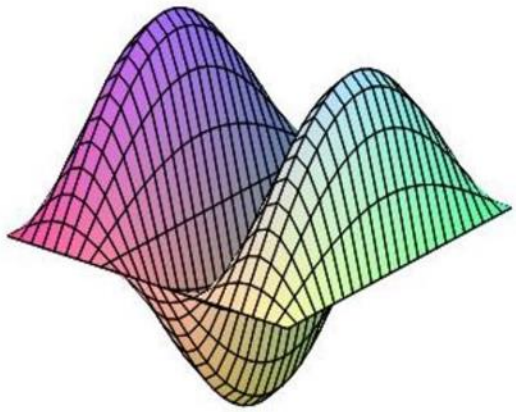
$\lambda_{21} = 168.05635479$is illustrated in Fig.2. and represented by the eigen function:

$$w_{21}(\zeta,\eta) = [1.0(10)^9\zeta + .5000000000\zeta^2 - 1.645483598(10)^9\zeta^3 + 3.200869939\zeta^4 + 2.628349946(10)^7\zeta^9 + 13.97522067\zeta^{10} - 2.271220655(10)^6\zeta^{11}][\sin 2\pi\eta = 0 \quad (29)$$

3.Third Mode m=3

The explicit eigen values $\lambda_{31}, \lambda_{32}, \lambda_{33}, \lambda_{34}, \ldots\ldots\ldots \lambda_{3n}$_3n are obtained from:

$$2.079942336(10)^{23} - 2.769390049(10)^{19}\lambda_2{}^2 + 1.388812471(10)^{15}\lambda_3{}^4 - 3.109683007(10)^{10}\lambda_3{}^6 + 2.622340201(10)^5\lambda_3{}^8 = 0 \quad (30)$$



In this case , the first three values of λ_3n are

$$\lambda_{31} = 374.7693805, \lambda_{32} = 380.9958646, \lambda_{33} = 381.4073682$$

Similarly, the eigen function corresponding to λ_31=374.7693805 is represented by the following equation and illustrated graphically in Fig.3.

Fig.3. Plate Third Mode Shape,

Fig.3.Plate Third Mode Shape, λ_31=374.7693805 , m=3,n=1

$$ww_{31}(x,y) = [1.0(10)^9\zeta + .5^*\zeta^2 - 1.643310542(10)^9\zeta^3 + 7.204425902\zeta^4 + 143.6624973\zeta^8 + 2.593850618(10)^7\zeta^9 + 291.7429516\zeta^{10} - 2.510964708(10)^6\zeta^{11}][\sin 3\,\pi\eta] = 0 \qquad (31)$$

4. CONCLUSION

The method of separating variables is presented in a new and improved form. Taking into account blended subordinates in the administering PDEs and direct replacements of limit conditions, numerical control turns out to be more troublesome. As a result, there are few publications of this kind in literature. The alternative numerical solutions don't require as much effort as the offered exact method does. The gave system decreases higher-request straight PDEs which integrate blended subordinates, into a bunch of effectively to deal with Tributes. Engineering problems governed by higher order linear PDE and involving mixed derivatives can be solved in a simple, direct, and highly accurate closed form by implementing the present method under specific boundary conditions. The legitimacy and unwavering quality of the strategy is analyzed through applying it to the instance of vibration of orthotropic plate in three modes. The application gives the arrangements in a precise scientific basic style.

CONFLICT OF INTERESTS

None.

ACKNOWLEDGMENTS

None.

REFERENCES

H. Ugail, Partial Differential Equations for Geometric Design, Springer Verlag, London, UK, 2011.

A. D. Polyanin, Handbook of Nonlinear Partial Differential Equations, 1st Edition, Chapman and Hall/CRC, New York, 2003.

T. Kobayashi, N. Kawashima and Y. Ochiai, Image processing by interpolation using polyharmonic function and increase in processing speed, IEEJ Transactions on Electrical and Electronic Engineering, 6(S1), S1-S6, 2011.

L. H. You, J. Changa, X. S. Yanga J. J. and Zhanga, Solid modelling based on sixth order partial differential equations. Computer-Aided Design, 43(6), 720-729, 2011.

M. H. Martin, A Generalization of the Method of Separation of Variables, Indiana Univ Math J, 2, 315-327, 1953.

E. Karimov and S. Pirnafasov. Higher Order Multi-Term Time- Fractional Partial Differential Equations Involving Caputo-Fabrizo Derivatives. Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 243, pp. 1-11.

I.V. Rakhmelevich, Multi-Dimensional Partial Differential Equations with Power Nonlinearities in the First Derivatives, Ufa Mathematical Journal. Vol. 9. No 1 (2016). P. 98-108.

W. N. Everitt and B. T. Johansson, Quasi-separation of the Biharmonic Partial Differential Equation, IMA Journal of Applied Mathematics (2009) 74, 85–709

A. S. Berdyshev and B. J. Kadirkulov , On a Nonlocal Problem for a Fourth-Order Parabolic Equation with the Fractional Dzhrbashyan–Nersesyan Operator, Differentially Uravneniya, 2016, Vol. 52, No. 1, pp. 123–127.

Xiang Liua , Xiao Liua and Suchao Xie . A highly accurate analytical spectral flexibility formulation for buckling and wrinkling of orthotropic rectangular plates. International Journal of Mechanical Sciences Volume 168,15 February 2020, 105311

A. H. A. Hassan and Naci Kurgan. Buckling of thin skew isotropic plate resting on Pasternak elastic foundation using extended Kantorovich method. www. elsevier.com/ locate/heliyon . Heliyon 6 (2020) e04236.

Xiang Liu, Xiao Liu and Wei Zhou. An analytical spectral stiffness method for buckling of rectangular plates on Winkler foundation subject to general boundary conditions. Applied Mathematical Modeling 86 (2020) 36–53.

Michael Doschoris, Towards a Generalization of the Separation of Variables Technique, Methods and Applications of Analysis. Vol. 19, No. 4, pp. 381–402, December 2012.

Travis Askham, A stabilized separation of variables method for the modified biharmonic equation, Journal of Scientific Computing, Vol.76, pp1674–1697(2018).

Hsieh M-C, Hwu C. Anisotropic elastic plates with holes/cracks/inclusions subjected to out-of-plane bending moments. Int J Solids Struct 2002;39(19): 4905-4905.