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MINIMUM ZERO-FORCING SETS OF VARIOUS GRAPHS: INSIGHTS AND REAL-WORLD APPLICATIONS

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ABSTRACT

Let each vertex of a graph G = (V (G), E(G)) be given one of two colours, say, “black” and “white”. Let Z denote the (initial) set of black vertices of G. The colour-change rule converts the colour of a vertex from white to black if the white vertex is the only white neighbour of a black vertex. Set Z is a zero forcing set of G if all vertices of G are turned black after finitely many applications of the colour-change rule. The zero forcing number of G is the minimum of |Z| overall zero forcing sets Z ⊂ V (G). This article explores the zero forcing in some graphs G. Additionally, it delves into the precise determination of minimum zero forcing set for certain well-known graphs.
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1. INTRODUCTION

The American Institute of Mathematics’ minimum rank special graph group has made significant strides in determining the zero forcing number for a graph G, denoted as Z(G). This graph parameter, Z(G), plays a crucial role in establishing bounds for the minimum rank across multiple families of graphs. In 2012, a pivotal study was conducted by Minerva Catral and her colleagues, who explored the intricate relationships between the maximum nullity, the zero forcing number, and the path cover number of graph G. Their research yielded valuable insights and results that have furthered the understanding of these graph parameters. They also examined how the zero forcing number correlates with the maximum nullity of edge subdivision graphs, adding depth to the existing body of knowledge. Building on these findings, in 2015, Amos and his team provided an upper bound for the zero forcing number of a connected graph G with n vertices, mainly focusing on cases where the maximum degree Δ is greater than or equal to 2. This work contributed critical information on the limitations and characteristics of zero-forcing numbers in connected graphs. Additionally, further noteworthy contributions in the field were made by Darren D. Row, who published a paper titled "A Technique for Computing Zero Forcing Number of a Graph with a Cut Vertex." In this work, Row introduced a methodology for calculating the zero forcing number specifically for graphs that contain cut vertices. His research spotlighted identifying graphs exhibiting extremely low or exceptionally high zero forcing numbers, enriching the ongoing discourse surrounding this intriguing graph parameter.

The paper discusses a type of graph colouring that defines a graph parameter called the zero forcing number, denoted by Z(G), which is the minimum size of the zero forcing set. This parameter was first introduced and defined in the Spectra of Families of Matrices workshop, described by graphs, digraphs, and sign patterns and held at the American Institute of Mathematics in October 2006. The paper investigates the zero forcing number of various graphs, an essential concept in graph theory. A zero forcing set is a subset of vertices in a graph that can influence the entire graph according to specific rules. Identifying these minimum sets is crucial for network theory and control systems applications. By examining different types of graphs, we can gain insights into the properties and utility of minimum zero forcing sets in various real-world contexts.

2. ZERO FORCING NUMBER

Definition 2.1. Let G be a graph with all vertices initially coloured, either black or white. If U is a black vertex of G and U has precisely one neighbour that is white, say v, then change the colour of v to black; this rule is called the colour change rule. In this case, we say,” u forces v”, denoted by $u \rightarrow v$. This procedure of colouring a graph using the colour change rule is called the zero forcing process. Note that each vertex will force one other vertex at most.

Given an initial colouring of G, the derived set is the set of all black vertices resulting from repeatedly applying the colour change rule until no more changes are possible.

Definition 2.2. [2] A zero forcing set, Z is a subset of vertices of G such that if initially the vertices in Z are coloured black and the **remaining** vertices are white, then the derived set of G is V (G). The zero forcing number of a graph G, denoted by Z(G), is the minimum size of a zero forcing set of G. Abbreviate the term zero forcing set as ZF S. A zero-forcing process called minimal if the initial set of black vertices is a minimal ZF S. Note that for any non-empty graph G, $1 \le Z(G) \le |V(G)| - 1$

Definition 2.3. [2] Let Z be a zero forcing set of a graph G. Construct the derived set, making a list of forces in the order in which they are performed. This list is called a chronological list of forces. A forcing chain is a sequence of vertices (v1, v2, ..., vk) such that vi → vi+1 for i = 1, 2, ..., k − 1.

Note that a minimal zero forcing process produces minimal forcing chains. A maximal forcing chain is a forcing chain that is not a proper subsequence of another zero forcing chain. Note that a zero-forcing chain can consist of a single vertex v1, and such a chain is maximal if v1 ∈ Z and v1 do not perform a force. In each step of a forcing process, each vertex can force at most one other vertex and can be forced by at most one other vertex. Therefore, the maximal forcing chain is disjoint. Thus, the vertices of the zero-force set the partition of the graph's vertices into disjoint paths. The number of chains in a zero forcing process starting with a zero forcing set Z is equal to the size of Z, and the elements of Z are the initial vertices of the forcing chains.

Let Z be a zero forcing set of a graph G. A reversal of Z is the set of last vertices of maximal zero forcing chains of a chronological list of forces. Thus, the cardinality of a reversal of Z is the same as that of Z.

Result 2.4. [2] If Z is a zero forcing set of G, so is any reversal of Z.

Result 2.5. [4] No connected graph of order is more significant than one with a unique minimum zero forcing set.

Result 2.6. [4] If G is a connected graph of order greater than one, then$\bigcap_{Z \in ZFS(G)} Z = \emptyset$

Definition 2.7. [2] Given a graph G = (V, E), a path cover is a set of disjoint induced paths in G such that every vertex v ∈ V belongs to exactly one path. The path cover number P (G) is the minimum number of paths in the path cover.

Lemma 2.8. [2] Let G be a graph. If $\chi \subset V(G)$ is a zero forcing set for G, χ induces a path cover for G.

Proof. It suffices to show that if H is the subgraph of G induced by the vertices of a zero forcing chain, then H is a path. By construction, H must contain a path. Suppose that H is not a path. Then either H is a cycle, or H has a vertex w of degree more than two. If H is a cycle with more than one vertex, then the first vertex u (the vertex of H that appears first in the zero forcing chain) has degree 2 in H, meaning that u has 2 white neighbours and cannot force anything. On the other hand, if some vertex w in H has degree at least 3, then w always has at least 2 white neighbours. In either case, we have a contradiction. □

Corollary 2.9. [2] For any graph G, $Z(G) \geq P(G)$.

Definition 2.10. [2] A path cover P of G is a Z-induced path cover of G if P can be induced by some zero forcing set of G. A minimal Z-induced path cover is a Z-induced path cover associated with a minimal zero forcing set.

A set X is a zero forcing set if and only if there is a z-induced path cover of G associated with X.

Observation 2.11. [2] Any superset of a zero forcing set for a graph G is also a zero forcing set for G. Equivalently if P∈ is a path cover of G obtained from a Z-induced path cover P∞ by splitting one or more paths in P∞, then P∈ is also a Z-induced path cover for G.

Result 2.12. For any graph G, $Z(G) \geq \omega(G) - 1$

Definition 2.13. [2] A vertex v in graph G is considered Z-terminal if it occurs as an endpoint in some minimal Z-induced path cover.

Corollary 2.14. [2]

The following statements are equivalent

(1) A vertex v is Z-terminal.

(2) There exists a minimal zero forcing set containing v.

(3) There is a minimal forcing set for which v is the last point in a maximal forcing chain.

Example 2.15. A pendent vertex is always Z-terminal. It must appear as an endpoint of any path cover, specifically, as an endpoint in a Z-induced path cover. This means that when looking for a minimal zero forcing set, you can always choose a pendent vertex to be in that set.

Definition 2.16. [2] A vertex v is doubly Z-terminal if there exists some minimal Z-induced path cover for which v appears in a path of length 1.

Equivalently, there exists some minimal zero forcing set containing v, with a forcing sequence such that $v \to u$ does not occur for any u. v is simply Z-terminal if v is Z-terminal but not doubly Z-terminal.

Example 2.17. Isolated vertices are always doubly Z-terminal. The pendent vertices in Pn, $n \geq 1$, are simply Z-terminal.

Definition 2.18. [5] A path in a graph G is an open walk in which no vertex and, therefore, no edges are repeated.

Observation 2.19. [1] For any connected graph G = (V, E), Z = 1 if and only if G = Pn for some $n \geq 2$.

From the observation, it is clear that zero forcing number of paths, Z(Pn) =1.

Observation 2.20. Let G be a graph then, we have $Z(G) \geq \delta(G)$, where $\delta(G)$ denotes the minimum degree of the graph G.

Observation 2.21. Let G be a connected graph with $Z(G) = k = \delta(G)$ and let $\hat{G}$ be the graph obtained from G by adding a single vertex v and joining the vertex v to all vertices of G. Then, $Z(\hat{G}) = Z(G) + 1$.

Definition 2.22. [5] A cycle in a graph is a non-empty trail in which only the first and last vertices are equal. From the above observations, it is clear that zero forcing number of cycles, Z(C) = 2.

Definition 2.23. A wheel graph is a graph formed by connecting a single universal vertex to all vertices of a cycle.

We can easily verify that if G is a wheel graph, then Z(G) = 3.

Definition 2.24. The ladder graph Ln is the graph obtained by taking the cartesian product of Pn and P2.

Observation 2.25. The graph Ps cartesian product with Pt and $Z = \{(1, j) : 1 \leq j \leq t\} \cup \{(i, 1): 1 \leq i \leq s\}$ is a zero forcing set. Thus $Z(Ps \boxtimes Pt) \leq s + t - 1$.

From the observation, it was proved that if G is the ladder graph, then Z(G) = 2.

Definition 2.26. A connected acyclic graph is called a tree.

Proposition 2.27. [2] For any tree T, P (T) = Z (T). Moreover, any minimal path covering of a tree P (T) coincides with a collection of forcing chains with |P (T)| = Z(G) and the set consisting of one endpoint from each path in P (T) is a ZF S for T.

Proof. We prove this by induction on the path cover number. The theorem applies for any tree with P (T) = 1 (a path). To perform the induction step, we need to prove the following claim. Claim: In any minimal path covering of a tree, there is always a path connected (through an edge) to only one other path in the path covering. We call such a path a pendent path.

To observe this, suppose there is no such path in a minimal path covering of a tree T. Thus, any path is connected to at least two other paths in the path covering. This means the graph has a cycle as a subgraph, which contradicts T being a tree.

Assume the theorem holds for all trees T' with $P(T') < P(T)$. Let P (T) be a path covering of T with |P(T)| = P (T).Let Z be the set consisting of one end-point of each path in P(T) and P1 be a pendent path in P(T) that is joined to the rest of T by only one edge uv with $v \in V(P1)$ and $u \notin V(P1)$.

Then, by repeatedly applying the colour-change rule starting at the black endpoint of P1, all vertices from the black endpoint through v are coloured black. Now the path P1 is irrelevant to the analysis of the tree T −V (P1), thus by the induction hypothesis, the black endpoints of the remaining paths are a zero forcing set for T −V (P1), and all vertices not in P1, including u, can be coloured black. Hence, the remainder of path P1 can also be coloured black, and Z is a zero forcing set for T. Moreover, all the forces are performed along the paths in P(T), which completes the proof. □

Definition 2.28. [5] A complete graph is a graph in which each vertex is connected to every other vertex. That is, a complete graph is an undirected graph, where every pair of distinct vertices is connected by a unique edge.

Proposition 2.29. [1] Let Kn be a complete graph. Then Z(Kn) = n − 1.

Definition 2.30. A complete bipartite graph is a special type of bipartite graph where every vertex of one set is connected to every vertex of another set.

Proposition 2.31. [4] Let Km,n be a complete bipartite with m, n ≥ 2.

Then Z(Km,n) = m + n − 2.

Proof. Let X and Y be the bipartite sets of Km,n with |X| = m and |Y | = n.

We will assume without loss of generality that m ≥ n.

Suppose we label the vertices in X as x1, x2, . . . ,xm and the vertices in Y as y1, y2, . . . , yn. If we color all vertices except x1 and y1 black, then by the basic colour change rule x2 will forces y1 black because that is the only white vertex adjacent to x2 in Y. Similarly, y2 will force x1 black. Hence Z(Km,n) ≤ m + n − 2.

To show Z(Km,n) is not less than m + n − 2 , we consider colouring all but three vertices of Km,n black.

Case 1: Suppose any three vertices in X are coloured white and the rest of the vertices in X are coloured black and all vertices of Y are coloured black. Then by the basic colour change rule, the black vertices in Y cannot force all three of the white vertices in X black.

Case 2: Suppose any two vertices in X are coloured white and the rest of the vertices in X are coloured black and a single vertex in Y is coloured white and the rest of vertices in Y are coloured black. A black vertex in X will force the white vertex in Y black by basic colour change rule. But the black vertices in Y cannot force both white vertices in X black. Since the other (two) cases are similar we conclude that Z(Km,n) > m +n − 3 . Hence Z(Km,n) = m + n − 2. □

Definition 2.32. Star graph is a special type of graph in which n−1 vertices have degree one and a single vertex have degree n − 1.

From the zero forcing number of complete bipartite graph, we can easily verify that zero forcing number of star graph, Z (K1, n) = 1 + n − 2 = n − 1.

3. CONCLUDING REMARKS

Zero forcing sets are crucial in estimating a matrix's minimum rank associated with a given graph. This concept is particularly important as it has significant implications for various areas, such as matrix completion problems and data analysis. Understanding zero forcing sets helps determine how we can efficiently fill in missing entries in matrices, which aids in better interpretation and utilisation of data in different analytical contexts. In addition, it can help monitor power systems. By representing the power grid infrastructure as a graph, we can utilise zero forcing sets to identify the most effective locations for placing sensors, known as phase measurement units. These strategically positioned sensors are crucial for comprehensive monitoring of the entire system, enabling us to track its performance and ensure stability.

This approach enhances our understanding of the grid's operation and optimises the deployment of monitoring resources to cover all critical areas efficiently.

The use of it for quantum control is undisputed. Zero forcing is a robust mathematical technique that can be employed to develop effective control strategies for quantum systems. This approach's primary objective is to precisely manipulate the system's quantum state by selectively applying control operations to designated qubits. This allows for tailored interventions that can steer the quantum system toward desired states or outcomes while taking advantage of the unique properties of quantum mechanics. By strategically targeting specific qubits, zero forcing enhances the control and efficiency of quantum operations in complex systems. It can also aid in sensor network design. By conceptualising sensors as vertices within a graph structure, zero forcing analysis is a powerful tool to identify the most miniature possible set of sensors required to monitor an entire area effectively. This method leverages the relationships between the sensors, allowing for an efficient deployment strategy that maximises coverage while minimising resources. Finally, it aids graph searching problems. Zero forcing is a concept that intersects with graph-searching algorithms. It focuses on "capturing" all the vertices in each graph. This is achieved by strategically manoeuvring a "searcher" throughout the graph's structure. The aim is to ensure that, through these movements, every vertex is ultimately dominated or influenced, leading to the complete coverage of the graph. The strategic placement and movement of the searcher are crucial for efficiently traversing the graph and achieving the objective of total vertex capture.

In short, the theory of zero forcing is a significant area of study that has garnered substantial attention from researchers, resulting in a vast body of published papers exploring its intricacies and applications. The zero forcing number, a critical concept within this theory, has proven highly versatile and is found to be helpful in various fields. For instance, it is essential in coding theory, aiding error detection and correction. In logic circuits, it is instrumental in optimising circuit designs. Additionally, the theory is applied in modelling the spread of diseases, helping researchers understand and predict outbreaks. Power network monitoring provides insights into the flow of electricity and system stability. Lastly, within social networks, the zero forcing concept facilitates the analysis of information dissemination and connectivity among users. Overall, the multifaceted applications of zero forcing illustrate its importance across numerous disciplines.

CONFLICT OF INTERESTS

None.

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