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MODIFIED ANALYTICAL METHODS FOR SOLVING FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

Fractional partial differential equations (FPDEs) extend classical PDEs by incorporating fractional order derivatives, capturing complex phenomena in various fields such as physics, engineering, and finance. Traditional analytical methods often fall short in efficiently solving FPDEs due to their inherent complexity and non-local behavior. This paper presents modified analytical methods designed to address these challenges. By integrating fractional calculus with advanced techniques such as operational matrices and modified Laplace transforms, the proposed methods offer improved accuracy and computational efficiency. The paper demonstrates the efficacy of these approaches through a series of numerical examples and comparative analyses with existing methods. The results highlight significant advancements in the solution of FPDEs, offering a robust framework for future research and practical applications.

Keywords: Fractional Partial Differential Equations (FPDEs), Fractional Calculus, Analytical Methods, Operational Matrices

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1. INTRODUCTION

Fractional partial differential equations (FPDEs) are increasingly recognized for their ability to model complex systems where classical differential equations fall short. These equations involve derivatives of non-integer order, providing a more nuanced representation of phenomena characterized by memory effects and non-local interactions. However, the application of traditional analytical techniques to FPDEs presents substantial challenges due to their complexity and the non-local nature of fractional derivatives. Existing methods often struggle with issues of accuracy and computational efficiency. This paper aims to overcome these limitations by proposing modified analytical methods that leverage advances in fractional calculus and numerical techniques. By integrating approaches such as operational matrices and modified Laplace transforms, the proposed methods enhance both the precision and practicality of solving FPDEs, potentially broadening their applicability across various scientific and engineering domains.

1.1 NEW MODIFICATION OF ADM FOR SOLVING NONLINEAR SYSTEMS OF FPDES

This section presents a novel and reliable ADM variation for solving nonlinear systems of first-order FPDEs (with the following form of starting values) that are presented in this section: (Saha Ray, S. 2021)

$$\begin{cases} \mathcal{D}_t^{q_i} u_i(\bar{x},t) = f_i(\bar{x},t) + L_i\bar{u}(\bar{x},t) + N_i\bar{u}(\bar{x},t), \; m_i - 1 < q_i \leq m_i \in \mathbb{N}, \\ \dfrac{\partial^{k_i} u_i(\bar{x},0)}{\partial t^{k_i}} = f_{ik_i}(\bar{x}), \; k_i = 0,1,2,\ldots,m_i-1, \end{cases}$$

…………1.2.1

For $i = 1,2,\ldots,m$ and $\bar{u}(\bar{x},t) = (u_1(\bar{x},t), u_2(\bar{x},t), \ldots, u_m(\bar{x},t)), \; \bar{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ in which the linear operator Li and the nonlinear operator Ni are defined as $\bar{u} = \bar{u}(\bar{x},t)$It may involve further fractional derivatives, the partial derivatives of it, and $f_{ik_i}(\bar{x}), f_i(\bar{x},t)$ well-established analytical operations and $\mathcal{D}_t^{q_i}$ determine the orders qi's The partial derivatives of Caputo's time function. Should the unexpected $f_i(\bar{x},t) = 0,$, 1.2.1, the system's homogeneous form, is realized. (Vahidi, J. 2021)

We take it as read that the answers $u_i(\bar{x},t)$ these analytic extensions are applicable to (1.2.1):

$$u_i(\bar{x},t) = \sum_{k=0}^{\infty} u_{ik}(\bar{x},t), \; i = 1,2,\ldots,m.$$

……………………………………………1.2.2

It was presumed that the functions would be affected by the updated decomposition approach. $u_{i0}(\bar{x},t)$ (for $i = 1,2,\ldots,m$) can be broken down into its component elements, namely $\phi_{i1}(\bar{x},t)$ and $\phi_{i2}(\bar{x},t)$ as well as being expressed as:

$$u_{i0}(\bar{x},t) = \phi_{i1}(\bar{x},t) + \phi_{i2}(\bar{x},t), \; i = 1,2,\ldots,m.$$

…………………………………..1.2.3

The important distinction is that in this case, just one component, namely, $\phi_{i1}(\bar{x},t)$ is going to be given to the component that comes last $u_{i0}(\bar{x},t),$ as opposed to the rest

$\phi_{i2}(\bar{x},t)$ when used in conjunction with the other words to describe $u_{i1}(\bar{x},t).$

Definition 1.2.1. The following is the definition of Ain: Ain(ui0,ui1,...,uin) for i = 1, 2,..., m.:

$$A_{in}(u_{i0}, u_{i1}, \ldots, u_{in}) = \frac{1}{n!}\frac{\partial^n}{\partial \lambda^n}\left[N_i \sum_{k=0}^{n} \lambda^k \bar{u}_k\right]_{\lambda=0}, \; i = 1,2,\ldots,m.$$

………………..1.2.4

We call $A_{in}(u_{i0}, u_{i1}, \ldots, u_{in})$ for example, generalized Adomian polynomials.

Remark 1.1. Let $A_{in} = A_{in}(u_{i0}, u_{i1}, \ldots, u_{in})$. Definition 1.2.1 and Theorem 1.3.1 are used to define the nonlinear operators. $N_i\bar{u}$ for $i = 1,2,\ldots,m,$has when expressed in terms of Admomian polynomials, this form follows: (Iğci, N. 2021)

$$N_i \sum_{k=0}^{\infty} \bar{u}_k = \sum_{n=0}^{\infty} A_{in}, \; i = 1,2,\ldots,m.$$

…………………………………………1.2.5

Theorem 1.2.1 (Existence Theorem). Let $m_i - 1 < q_i < m_i \in \mathbb{N}, \, for \; i = 1,2,\ldots m,$ and let $f_i(\bar{x},t), f_i(\bar{x})$ the analytical functions that are used. Then, solutions provided by can be accepted by the system (4.2.1).

$$u_i(\bar{x},t) = f_{it}^{(-q_i)}(\bar{x},t) + \sum_{k_i=0}^{m_i-1} \frac{t^{k_i}}{k_i!} f_{ik_i}(\bar{x}) + \sum_{k=2}^{\infty} [L_{it}^{(-q_i)} \bar{u}_{(k-1)} + A_{i(k-1)t}^{(-q_i)}],$$ ………….1.2.6

for i = 1,2,...m, where $f_{it}^{(-q_i)}(\bar{x},t),\ L_i^{(-q_i)} \bar{u}_{(k-1)t}\ and\ A_{i(k-1)t}^{(-q_i)} = \mathfrak{I}_t^{q_i} A_{i(k-1)}(u_{i0}, u_{i1}, \dots, u_{i(k-1)})$ denote to q_i^{th} integral for time, fraction, and partial $f_i(\bar{x},t),\ L_i \bar{u}_{k-1}\ and\ A_{i(k-1)}(u_{i0}, u_{i1}, \dots, u_{i(k-1)})$ respectively.

Proof.By solving the system's equations (1.2.1) using the partial integral of the Riemann-Liouville time function with fractions we are able to verify Theorem 1.2.1. (Fernandez, A. 2021)

$$u_i(\bar{x},t) = f_{it}^{(-q_i)}(\bar{x},t) + \sum_{k_i=0}^{m_i-1} \frac{t^{k_i}}{k_i!} \frac{\partial^{k_i} u_i(x,0)}{\partial t^{k_i}} + \mathfrak{I}_t^{q_i}[L_i \bar{u}] + \mathfrak{I}_t^{q_i}[N_i \bar{u}],$$ …………………1.2.7

when i ranges from 1 to m. It is possible to obtain by transferring the starting condition from system (1.2.1) to system (1.2.7):

$$u_i(\bar{x},t) = f_{it}^{(-q_i)}(\bar{x},t) + \sum_{k_i=0}^{m_i-1} \frac{t^{k_i}}{k_i!} f_{ik_i}(\bar{x}) + \mathfrak{I}_t^{q_i}[L_i \bar{u}] + \mathfrak{I}_t^{q_i}[N_i \bar{u}],\ i = 1,2,\dots,m.$$ ……….1.2.8

Using the decomposition technique, the unknown functions are broken down.

$u_{ik}(\bar{x},t)$ fractioned into their individual components by applying the decomposition series given by equation (1.2.2). So, we get by replacing system (4.2.2) with (4.2.8):

$$\sum_{k=0}^{\infty} u_{ik}(\bar{x},t) = f_{it}^{(-q_i)}(\bar{x},t) + \sum_{k_i=0}^{m_i-1} \frac{t^{k_i}}{k!} f_{ik_i}(\bar{x}) + \mathfrak{I}_t^{q_i}[L_i \sum_{k=0}^{\infty} \bar{u}_k] + \mathfrak{I}_t^{q_i}[N_i \sum_{k=0}^{\infty} \bar{u}_k],$$ ………..1.2.9

For i = 1,2,...,m. The linear terms $L_i \bar{u}$ satisfy

$$L_i \bar{u}(\bar{x},t) = L_i \sum_{k=0}^{\infty} \bar{u}_k(\bar{x},t) = \sum_{n=0}^{\infty} L_i \bar{u}_k(\bar{x},t),\ i = 1,2,\dots,m.$$ ………………………1.2.10

It is possible to rewrite system (1.2.9) as using Remark 4.1:

$$\sum_{k=0}^{\infty} \bar{u}_{ik}(x,t) = f_{it}^{(-q_i)}(\bar{x},t) + \sum_{k_i=0}^{m_i-1} \frac{t^{k_i}}{k_i!} f_{ik_i}(\bar{x}) + \mathfrak{I}_t^{q_i}[\sum_{k=0}^{\infty} L_i \bar{u}_k] + \mathfrak{I}_t^{q_i}[\sum_{n=0}^{\infty} A_{in}],$$ ………..1.2.11

when i ranges from 1 to m. We suggest a little tweak to in the event that changed ADM is really the case. $u_{i0}(\bar{x},t)$ and $u_{i1}(\bar{x},t)$ contrasted with ADM. Hence, it is presumed that $u_{0i}(\bar{x},t) = \phi_{i1}(\bar{x},t)$ for $i = 1,2,\dots,m,$ and here precisely what sets it apart is that $\phi_{i1}(\bar{x},t) = \sum_{k_i=0}^{m_i-1} \frac{t^{k_i}}{k_i!} f_{ik_i}(\bar{x})$ could be given to the 0th part $u_{i0}(\bar{x},t),$ contrasted with the rest $\phi_{i2}(\bar{x},t) = f_{it}^{(-q_i)}(\bar{x},t)$ is defined when used in conjunction with the other terms $u_{i1}(\bar{x},t).$

The following is how the revised recursive algorithm is formulated in light of these recommendations: (Khan, S. (2022)

$$\begin{cases} u_{i0}(\bar{x},t) = \sum_{k_i=0}^{m_i-1} \frac{t^{k_i}}{k_i!} f_{ik_i}(\bar{x}), \\ u_{i1}(\bar{x},t) = f_{it}^{(-q_i)}(\bar{x},t) + L_{it}^{(-q_i)} \bar{u}_0] + A_{i0t}^{(-q_i)}, \\ u_{ik}(\bar{x},t) = L_{it}^{(-q_i)} \bar{u}_{(k-1)} + A_{i(k-1)t}^{(-q_i)},\ k = 2,3,\dots,\ i = 1,2,\dots,m. \end{cases}$$ ……………..1.2.12

System (4.2.2) provides a decomposition series that can be expressed as:

$$u_i(\bar{x},t)=\sum_{k=0}^{n}u_{ik}(\bar{x},t)=u_{i0}(\bar{x},t)+u_{i1}(\bar{x},t)+\sum_{k=2}^{n}u_{ik}(\bar{x},t),\ i=1,2,\ldots,m. \quad \text{........1.2.13}$$

To finish the proof, we just need to insert the parts from (1.2.12) into (1.2.13).

Theorem 1.2.2 (Convergence Theorem). Prove that B is a Banach space. After that, a set of solutions $\{u_{in}(\bar{x},t)\}_{n=0}^{\infty}$ the value obtained by (1.2.2) approaches $S_i\in B\, for\, i=1,2,\ldots,m,$ if there exists $\gamma_i,\ 0\leq\gamma_i<1,\ then\ \|u_{in}\|\leq\gamma_i\|u_{i(n-1)}\|,\ \forall n\in\mathbb{N}.$

The evidence. The following are sequences of partial sums of series that determine Sin, as given in system (1.2.12):

$$\begin{cases} S_{i0}=u_{i0}(\bar{x},t),\\ S_{i1}=u_{i0}(\bar{x},t)+u_{i1}(\bar{x},t),\\ S_{i2}=u_{i0}(\bar{x},t)+u_{i1}(\bar{x},t)+u_{i2}(\bar{x},t),\\ \vdots\\ S_{in}=u_{i0}(\bar{x},t)+u_{i1}(\bar{x},t)+u_{i2}(\bar{x},t)+\cdots+u_{in}(\bar{x},t),\ i=1,2,\ldots,m. \end{cases} \quad \text{...............1.2.14}$$

We must then prove that $\{S_{in}\}_{n=0}^{\infty}$, in the Banach space B, with i = 1, 2,...., m, which are Cauchy sequences. With this objective in mind, we explorer (Mohamed, M. Z. 2021)

$$\|S_{i(n+1)}-S_{in}\|=\|u_{i(n+1)}(\bar{x},t)\|\leq\gamma_i\|u_{in}(\bar{x},t)\|\leq\gamma_i^2\|u_{i(n-1)}(\bar{x},t)\|\leq\cdots\leq\gamma_i^{n+1}\|u_{i0}\|, \quad \text{...1.2.15}$$

for i = 1,2,...,m. For every $n,r\in\mathbb{N},\ n\geq r,$ Through the sequential use of system (1.2.15) and triangle inequality, we are able to

$$\begin{aligned}\|S_{in}-S_{ir}\|&=\|(S_{in}-S_{i(n-1)})+(S_{i(n-1)}-S_{i(n-2)})+\cdots+(S_{i(r+1)}-S_{ir})\|\\ &\leq\|(S_{in}-S_{i(n-1)})\|+\|(S_{i(n-1)}-S_{i(n-2)})\|+\cdots+\|(S_{i(r+1)}-S_{ir})\|\\ &\leq\gamma_i^n\|u_{i0}(\bar{x},t)\|+\gamma_i^{n-1}\|u_{i0}(\bar{x},t)\|+\cdots+\gamma_i^{r+1}\|u_{i0}(\bar{x},t)\|\end{aligned}$$

$$\leq\gamma_i^{r+1}(1+\gamma_i+\cdots+\gamma_i^{n-r-1})\|u_{i0}(\bar{x},t)\|\leq\frac{\gamma_i^{r+1}(1-\gamma_i^{n-r})}{1-\gamma_i}\|u_{i0}(\bar{x},t)\|. \quad \text{.............1.2.16}$$

Since $0<\gamma_i<1,\ \text{so}\ 1-\gamma_i^{n-r}\leq1.$ Then

$$\|S_{in}-S_{ir}\|\leq\frac{\gamma_i^{r+1}}{1-\gamma_i}\|u_{i0}(\bar{x},t)\|,\ i=1,2,\ldots,m. \quad \text{.....................................1.2.17}$$

But $u_{i0}(\bar{x},t)$ has a finite value and we get

$$\lim_{n,r\to\infty}\|S_{in}-S_{ir}\|=0,\ i=1,2,\ldots,m. \quad \text{...1.2.18}$$

Consequently, the orders $\{S_{in}\}_{n=0}^{\infty}$, The solutions given by system (1.2.13) converge as Cauchy sequences in a Banach space B for all integers i = 1, 2,...., m.

Theorem 1.2.3 (Error Analysis). The most severe truncation mistakes in the solution series (4.2.6) for system (4.2.1) are anticipated to be

$$\sup_{(\bar{x},t)\in\Omega} |u_i(\bar{x},t) - \sum_{k=0}^{r} u_{ik}(\bar{x},t)| \leq \frac{\gamma_i^{r+1}}{1-\gamma_i} \sup_{(\bar{x},t)\in\Omega} |u_{i0}(\bar{x},t)|,\ \Omega \subset \mathbb{R}^{n+1},\ i = 1,2,\ldots,m.$$

……….1.2.19

The evidence. Based on Theorem 4.2.2, we may infer that

$$\|S_{in} - S_{ir}\| \leq \frac{\gamma_i^{r+1}}{1-\gamma_i} \sup_{(\bar{x},t)\in\Omega} |u_{i0}(\bar{x},t)|,\ i = 1,2,\ldots,m.$$

…………………………….1.2.20

Yet, it is believed that $S_{in} = \sum_{k=0}^{n} u_{ik}(\bar{x},t)$ from 1 to m, and because n approaches infinity, we get $S_{in} \to u_i(\bar{x},t).$ A possible rewriting of system (1.2.20) is as follows:

$$\|u_i(\bar{x},t) - S_{ir}\| = \|u_i(\bar{x},t) - \sum_{k=0}^{r} u_{ik}(\bar{x},t)\|$$

$$\leq \frac{\gamma_i^{r+1}}{1-\gamma_i} \sup_{(\bar{x},t)\in\Omega} |u_{i0}(\bar{x},t)|,\ i = 1,2,\ldots,m.$$

……………………………………….1.2.21

Thus, the worst possible absolute truncation mistakes in the set Ω, where i = 1, 2,..., m, are

$$\sup_{(\bar{x},t)\in\Omega} |u_i(\bar{x},t) - \sum_{k=0}^{r} u_{ik}(\bar{x},t)| \leq \frac{\gamma_i^{r+1}}{1-\gamma_i} \sup_{(\bar{x},t)\in\Omega} |u_{i0}(\bar{x},t)|,\ i = 1,2,\ldots,m,$$

…………1.2.22

as well as the proof being finished.

1.2 DISCUSSION AND NUMERICAL RESULTS

Table 1.1 displays the numerical comparison of the several values of x,t in Example 1.2.1.1, where q = 0.5, q = 1, and α = H = 0.5. It also includes the absolute error and both approximate and accurate answers. Figure 4.2 displays the numerical values of the exact and approximate solutions for Example 1.2.1.2 for various values of x and t for q1 = q2 = 0.5 and 1, respectively. This makes it possible to compare the two data sets. Fig. 4.1a displays the graphs of the approximate solutions for Example 1.2.1.1 when α = H = 0.5 and q = 1. The graphs corresponding to the precise results for Example 1.2.1.1 for α = β = 0.5 are shown in Figure 1.1b. The approximate solutions to Example 1.2.1.2 are presented in Fig. 4.2a with q1 = q2 = 1. The exact solution graphs for Example 1.2.1.2 are shown in Figure 1.2b.

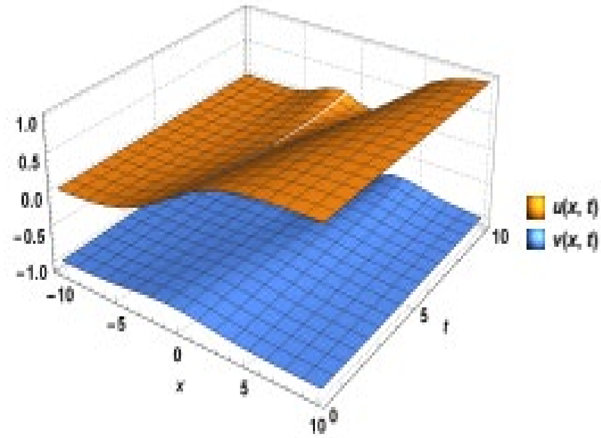
Table 1.1:For Example 1.2.1.1, the numerical values of the precise and approximate solutions for q = 0.5,1 and α = β = 0.5.

$\alpha=\beta=q=0.50$				$\alpha=\beta=0.5,\ q=1$		$\alpha=\beta=0.5$		Absolute Error at $q=1$	
x	t	$u(x,t)$	$v(x,t)$	$u(x,t)$	$v(x,t)$	$u_{EX}(x,t)$	$v_{EX}(x,t)$	$\lvert u_{EX}(x,t)-u(x,t)\rvert$	$\lvert v_{EX}(x,t)-v(x,t)\rvert$
	0.20	0.621909	-0.882706	0.639916	-0.884788	0.639916	-0.884788	3.06513×10^{-8}	1.16414×10^{-8}
0.25	0.40	0.609298	-0.881558	0.628316	-0.883233	0.628316	-0.883233	4.86441×10^{-7}	1.94750×10^{-7}
	0.60	0.599481	-0.880867	0.616564	-0.881795	0.616567	-0.881794	2.44103×10^{-6}	1.02872×10^{-6}
	0.20	0.650758	-0.886618	0.668188	-0.889144	0.668188	-0.889144	3.27352×10^{-8}	5.06759×10^{-9}
0.50	0.40	0.638477	-0.885130	0.657010	-0.887326	0.657010	-0.887326	5.21894×10^{-7}	8.93617×10^{-8}
	0.60	0.628869	-0.884155	0.645654	-0.885608	0.645656	-0.885608	2.63121×10^{-6}	4.94616×10^{-7}
	0.20	0.678547	-0.891172	0.695297	-0.894070	0.695297	-0.894070	3.32246×10^{-8}	1.03626×10^{-9}
0.75	0.40	0.666676	-0.889385	0.684601	-0.892039	0.684602	-0.892039	5.31703×10^{-7}	9.12339×10^{-9}
	0.60	0.106929	-0.952285	0.673704	-0.890087	0.673707	-0.890087	2.69101×10^{-6}	7.75541×10^{-9}

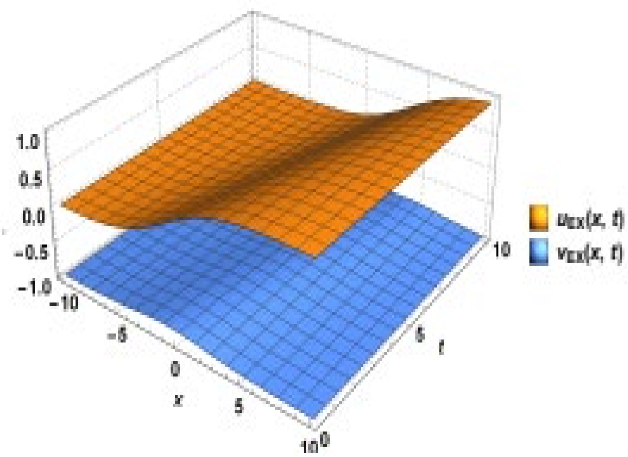
Table 1.2: For Example 1.2.1.2, the approximate and precise solutions' numerical values are given for q1 = q2 = 0.5 and q1 = q2 = 1.

$q_1=q_2=0.50$				$q_1=q_2=1$		Absolute Error at $q_1=q_2=1$			
x	t	$u(x,t)$	$v(x,t)$	$u(x,t)$	$v(x,t)$	$u_{EX}(x,t)$	$v_{EX}(x,t)$	$\lvert u_{EX}(x,t)-u(x,t)\rvert$	$\lvert v_{EX}(x,t)-v(x,t)\rvert$
	0.20	0.757584	1.331070	1.051190	0.951175	1.051270	0.951229	0.00008229	0.000054068
0.25	0.40	0.498626	1.656020	0.859441	1.160930	0.860708	1.161830	0.00126696	0.000901875
	0.60	0.229151	1.944980	0.698510	1.414300	0.704688	1.419070	0.00617826	0.004765330
	0.20	0.986646	1.025820	1.349750	0.740776	1.349860	0.740818	0.00010566	0.000042108
0.50	0.40	0.679531	1.259110	1.103540	0.904135	1.105170	0.904837	0.00162681	0.000702381
	0.60	0.366403	1.458550	0.896904	1.101460	0.904837	1.105170	0.00793305	0.003711240

	0.20	1.280770	0.788092	1.733120	0.576917	1.733250	0.576950	0.00013567	0.000032794
0.75	0.40	0.911817	0.950005	1.416980	0.704141	1.419070	0.704688	0.00208887	0.000547015
	0.60	0.542637	1.079710	1.151650	0.857818	1.161830	0.860708	0.01018620	0.002890320

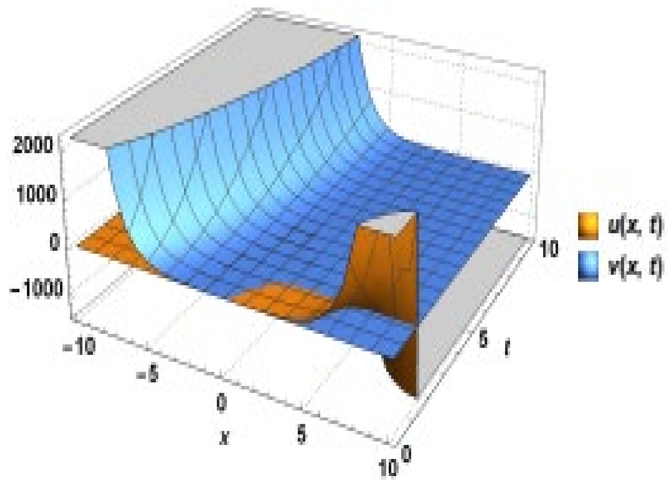


(a) The approximate solution graphs for u(x,t), v(x,t) for Example 4.2.1.1 when α = β = 0.5 and q = 1.

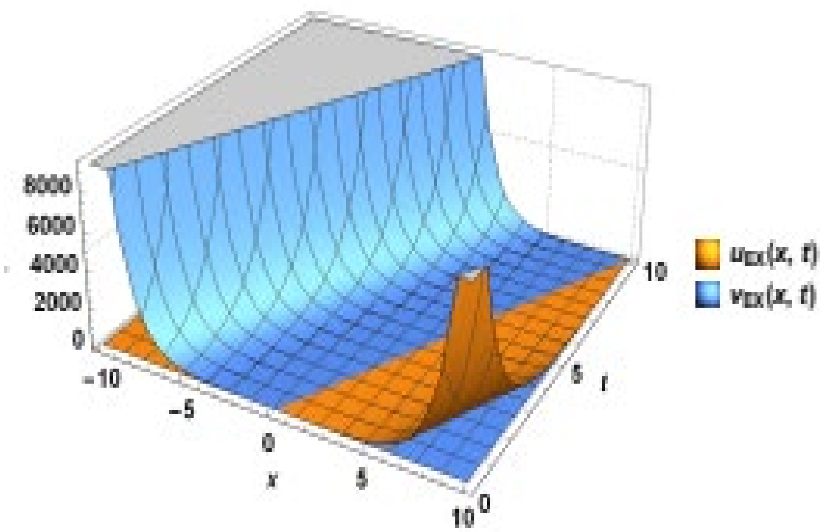


(b) For Example 1.2.1.1, the graphs show the precise solutions uEX (x,t) and vEX (x,t) when α = β = 0.5

Figure 1.1: The graphs representing the approximate and exact answers for Example 4.2.1.1 when α = β = 0.5 and q = 1.



(a) The approximate solution graphs for Example 4.2.1.2, u(x,t), and v(x,t), when q1 = q2 = 1. (El-Dib, Y. O. 2021)



(b) The graphs representing the precise answers for Example 4.2.1.2 are uEX (x,t) and vEX (x,t).

Figure 1.2: The graphs representing the precise and approximate answers for Example 4.2.1.2 when q1 = q2 = 1.

2. CONCLUSION

The modified analytical methods presented in this paper represent a significant advancement in solving fractional partial differential equations. By incorporating fractional calculus with innovative techniques such as operational matrices and modified Laplace transforms, these methods address the critical challenges of accuracy and computational efficiency that are prevalent in traditional approaches. The numerical results demonstrate that the proposed methods provide substantial improvements over existing solutions, offering a more robust and practical framework for tackling FPDEs. This advancement not only enhances theoretical understanding but also opens new avenues for applying FPDEs in real-world scenarios. Future research should explore further refinements and extensions of these methods, as well as their application to more complex and higher-dimensional problems.

CONFLICT OF INTERESTS

None.

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None.

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